10. Exercise sheet
Hand in solutions until Tuesday, 30 January, 12\(^{15}\).

**Exercise 10.1** (Experiment). (6+1 points)

Given a function \( h \) it is claimed that the formula
\[
f(h) = \frac{V(h)}{2\pi \alpha(h)}
\]
with
\[
V(h) = \frac{1}{2k} \int_{-k}^{k} |h'(x)| \, dx, \quad \text{(variation)}
\]
\[
\alpha(h) = \frac{1}{2k} \int_{-k}^{k} |h(x) - E(h)| \, dx \quad \text{(average amplitude), and}
\]
\[
E(h) = \frac{1}{2k} \int_{-k}^{k} h(x) \, dx \quad \text{(mean value)}
\]
gives the desired average frequency.

(i) Use functions \( h_f(x) = \cos(2\pi f \cdot x) \). Calculate (numerically) its average frequency \( f(h_f) \). Use \( k = 1 \) with \( f \in [0, 10] \) and compare \( f(h_f) \) with \( f \). (You might have problems if \( k \cdot f \geq 4 \ldots )

(ii) Take again the wave \( h(x) = \cos(2\pi f \cdot x) \). Derive the formula:
\[
\circ \quad \text{Now express } h'(x) \text{ as a multiple of a shift } h(x+\delta) \text{ of } h, \text{ so } h'(x) = \varphi \cdot h(x+\delta). \text{ Integrate absolute value of the formula over an interval } -k..k.
\]
\[
\circ \quad \text{Assuming that } k \gg \delta, \text{ we can shift the integration interval on the right by } \delta \text{ without losing much.}
\]

**Exercise 10.2** (Poincaré index). (6 points)

Choose random values for the direction in the cells around a given point. (Choose eight angles!) The direction here indicates the direction of an oriented ridge through that point. In some cases mark one (say, the first one) of them as ‘unorientable’, ie. we assume the value to be known only up to \( 180^\circ \). Compute the Poincaré index \( P = \sum_{0 \leq k < 8} \angle(d_k, d_{k+1}) \). Clearly, \( \angle(\varphi, \psi) \) is related to \( \varphi - \psi \) but is always between \( -180^\circ \) and \( 180^\circ \).

(i) Repeat the experiment some times and

(ii) discuss your results. (You might want to draw some pictures to interpret them.)

(Actually, this is somehow a discretization of \( \frac{1}{2\pi} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz \) which counts how often a continuous path \( \gamma \) loops around zeroes of a complex function \( f \).)