# Cryptography

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## 2. Assignment: Arithmetic in $\mathbb{F}_{256}[y]$

(Due: Wednesday, 12 November 2008, $13^{40}$, b-it bitmax)

**Exercise 2.1** (`MixColumns`). (12 points)

The `MixColumns`-step of the AES-algorithm takes place in the ring

$$S = \mathbb{F}_{256}[y]/\langle y^4 + 1\rangle.$$

(i) The ring $S$ is not a field. In particular, there are nonzero elements in $S$ *without* a multiplicative inverse. Give an example and explain how you could check that property. $\boxed{2}$

(ii) The output $b_3$, $b_2$, $b_1$ and $b_0$ of the `MixColumns`-step for a column with entries $a_3$, $a_2$, $a_1$ and $a_0$ is determined by the product $\boxed{3}$

$$b_3y^3 + b_2y^2 + b_1y + b_0 = (\texttt{02} + \texttt{01}y + \texttt{01}y^2 + \texttt{03}y^3) \cdot (a_3y^3 + a_2y^2 + a_1y + a_0).$$

Expand the product over $\mathbb{F}_{256}[y]$, reduce it modulo $y^4 + 1$ and collect the terms with equal powers of $y$ to obtain equations for $b_3$, $b_2$, $b_1$ and $b_0$.

(iii) Find a $4 \times 4$-matrix $\mathcal{M}$ with entries from $\mathbb{F}_{256}$ to express this multiplication as a matrix-vector product $\boxed{2}$

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \mathcal{M} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

(iv) Use this matrix-vector product to perform the `MixColumns`-operation on the following state of AES: $\boxed{3}$

$$\begin{bmatrix} \texttt{00} & \texttt{00} & \texttt{00} & \texttt{00} \\ \texttt{7A} & \texttt{00} & \texttt{00} & \texttt{00} \\ \texttt{01} & \texttt{00} & \texttt{01} & \texttt{00} \\ \texttt{00} & \texttt{00} & \texttt{00} & \texttt{AA} \end{bmatrix}$$

(v) The `InvMixColumns`-operation is the inverse of `MixColumns`. From exercise 1.3 (iii) you know that the product of $\texttt{02} + \texttt{01}y + \texttt{01}y^2 + \texttt{03}y^3$ with $\texttt{0B}y^3 + \texttt{0D}y^2 + \texttt{09}y + \texttt{0E}$ is $\texttt{01}$ in $S$. Use this information to write down the `InvMixColumns`-operation on a column $b$ in matrix-vector-notation. $\boxed{2}$

**Exercise 2.2** (`Repeated-Squaring`). (8 points)

For a given positive integer $N$ we consider the set $\mathbb{Z}_N$ of all residues modulo $N$, i.e.

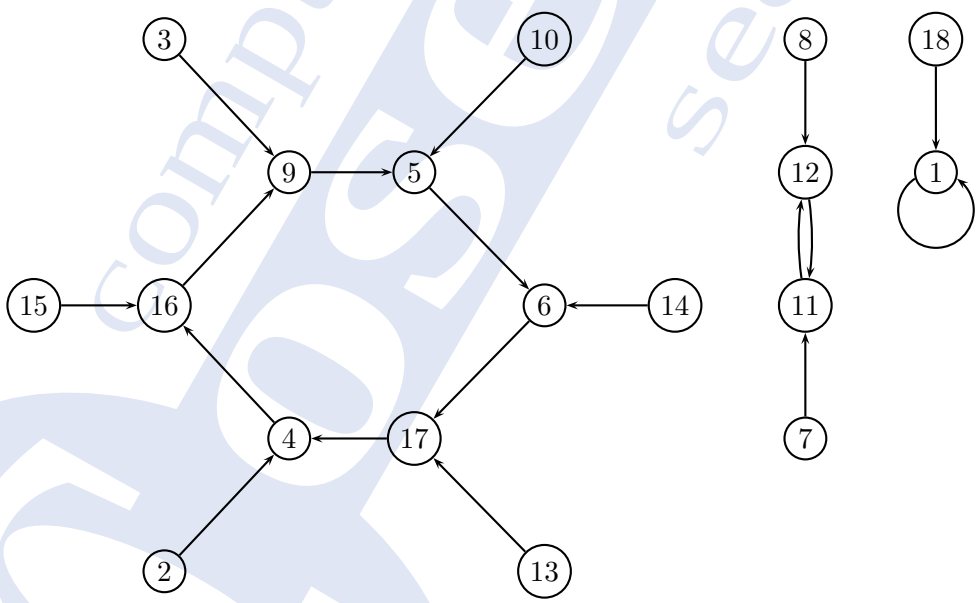
$$\mathbb{Z}_N = \{0, 1, \ldots, N-1\}.$$

On this set we can add, multiply and raise to powers, by simply performing the well-known corresponding operation on integers and reducing the result modulo $N$ if necessary. We want to examine the operation of `squaring` and visualize the relations that are generated by repeated squaring.

This can be done by drawing a directed graph according to the following recipe:

1. Draw $N-1$ vertices and label them $1, 2, \ldots, N-1$.
2. Draw an arrow from every vertex to the one labeled by its square modulo $N$.
3. Arrange your picture well.

3 (i) As an example, the graph for $N = 19$ consists of the following three components:




Use the information from this graph and the `Repeated-Squaring`-method to compute $15^{22} \mod 19$. How many multiplications are necessary?

4 (ii) Draw a graph for $N = 13$ and $N = 17$.

1 (iii) How many sources (i.e. vertices where no arrow is pointing to) does each of the graphs have.