Heads and Tails, summer 2009

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1. Exercise sheet
Hand in solutions until Sunday, 19 April 2009, 24:00h.

Reminders.

- For the course we remind you of the following dates:
  - Lectures: Monday and Thursday 13:00h-14:30h **sharp**, b-it bitmax.
  - Tutorial: Monday 14:45h-16:15h **sharp** also at the b-it, bitmax.
- A word on the exercises. They are important. Of course, you know that. Just as an additional motivation, you will get a bonus for the final exam if you attended the tutorial regularly and earned more than 60% or even more than 80% of the credits.

Exercise 1.1 (Expected value of a random variable). (9 points)

We are given a discrete random variable X, for example the result of a single roll of a fair die. The values that X can take are denoted by x and the respective probability is given by $\mathrm{prob}(X = x)$. For the example, the x are taken from the set $A = \{1, 2, 3, 4, 5, 6\}$ each with $\mathrm{prob}(X = x) = 1/6$.

We are interested in the *expected value* $E(X)$ defined as

$$E(X) = \sum_x x \cdot \mathrm{prob}(X = x),$$

where the sum is taken over all possible values of X. In the example above, this returns as the expected value for the roll of a single die

$$E(X) = \sum_{x \in A} x \cdot \frac{1}{6} = \frac{21}{6} = 3.5.$$

Next, we roll the die until a certain number, say "2", appears *for the first time*. The random variable Y is now the *number of rolls* that are performed, until this happens.

(i) What is $\mathrm{prob}(Y = i)$, i.e. the probability that "2" appears for the first time in the ith roll? 2

(ii) Prove that $E(Y) = 6$. (You may have use for the generalization of the formula for the geometric series $\sum_{k=n}^{\infty} q^k = q^n/(1-q)$ for $|q| < 1$.) 4

(iii) Generalize the preceding steps to prove the more general proposition 3

Proposition. *Suppose that an event A occurs in an experiment with probability p, and we repeat the experiment until A occurs. Then the expected number of executions until A happens is $1/p$.*