2. Exercise sheet
Hand in solutions until Sunday, 8 November 2009, 23\textsuperscript{59}

Exercise 2.1 (Characteristic). \hfill (3 points)

Let $k$ be a field. Recall that $\text{char}(k) := \min \{ m \mid m \cdot 1_k = 0 \}$ where we define $\min \{ \} := \infty$. Show that the characteristic $\text{char}(k)$ of $k$ is either prime or $\infty$.

Exercise 2.2 (Smoothy). \hfill (14 points)

Let $C$ be a cubic curve given by a polynomial $f \in k[x,y]$ and let $F \in k[X,Y,Z]$ be its homogenization, i.e. $F = Z^3 f(X/Z, Y/Z)$. In the course we saw that $f$ is smooth if and only if it is smooth at infinity and there are no (affine) solutions to the system of equations $f = 0, f_x = 0, f_y = 0$. We will show that this is equivalent to the requirement that the system of equations $F = 0, F_X = 0, F_Y = 0, F_Z = 0$ has only the solution $X = Y = Z = 0$. To do so let $P = (x, y)$ be a point on $\{ f = 0 \}$, that is with $f(P) = 0$.

(i) Show that if all derivatives of $f$ vanish at $P$ then all derivatives of $F$ vanish at $(X,Y,1)$, where $X = x$ and $Y = y$.

(ii) Show that for any homogeneous polynomial $F$ all derivatives are homogeneous.

(iii) Show that we always have

$$F(0,0,0) = F_X(0,0,0) = F_Y(0,0,0) = F_Z(0,0,0) = 0.$$  \hfill 1

(iv) Using (ii), show that if all derivatives of $F$ vanish at $(X,Y,1)$ then all derivatives of $F$ vanish at $(\beta X, \beta Y, \beta)$ for some $\beta \in k^\times$.

(v) Conclude that if all derivatives of $F$ vanish at $(X,Y,Z)$ then

- all derivatives of $f(U,V) = F(U,V,1)$ vanish at the point $\left( \frac{X}{Z}, \frac{Y}{Z} \right)$ provided $Z \neq 0$,
- all derivatives of $g(U,V) := F(U,1,V)$ vanish at the point $\left( \frac{X}{Y}, \frac{Z}{Y} \right)$ provided $Y \neq 0$,
- all derivatives of $h(U,V) := F(1,U,V)$ vanish at the point $\left( \frac{Y}{X}, \frac{Z}{X} \right)$ provided $X \neq 0$.  \hfill 3
(vi) We have now shown that for a single point \( P = X : Y : Z \) with \( XYZ \neq 0 \) we can either look at \( f \) and its derivatives or at \( g \) or at \( h \). For a point where one of the coordinates vanishes, we look at those functions among \( \{f, g, h\} \) which pose no division problem. Prove now that the curve is smooth at all its points iff the system of equations \( F = 0, F_X = 0, F_Y = 0, F_Z = 0 \) has only the solution \( X = Y = Z = 0 \).

Exercise 2.3 (Transformers). (8 points)

Consider the curve given by the polynomial \( x^3 - 2xy^2 + y^3 - x \) over the reals.

(i) Show that this curve has three points at infinity.

(ii) Write down the transformation that maps the (projective point representing the) \( x \)-direction to \((0, 0)\), the \( y \)-direction to the \( x \)-direction, and the point \((0, 0)\) to the \( y \)-direction.

(iii) Compute the equation of the transformed curve.

(iv) Plot the curve and its transform.

Exercise 2.4 (Degree? Invariant.). (9+4 points)

In the lecture we have discussed projective linear transformations. Namely we consider a transformation \( \tau: \mathbb{P}k^2 \rightarrow \mathbb{P}k^2, \ x \mapsto M \cdot x \), where \( M \in k^{3 \times 3} \) is invertible.

(i) Consider an affine point \((x, y)\). We apply \( \tau \) by first embedding \((x, y)\) in the projective space, obtaining the point \((X : Y : 1)\). We now apply \( \tau \), obtaining \((U : V : W) := \tau(X : Y : Z)\). If \( W \) is non-zero we get the transformed point \((u, v) = (U/W, V/W)\). Combine all those steps and write down the resulting map \( \sigma: k^2 \rightarrow k^2, \ (x, y) \mapsto (u, v) \), i.e. give the corresponding formulas for \( u \) and \( v \).

(ii) Show that \( \sigma \) maps lines to lines.

(iii) Show that \( \sigma \) maps ellipses, hyperbolas and parabolas to ellipses, hyperbolas and parabolas.

(iv) Generalize this to show that \( \sigma \) maps curves defined by a polynomial \( g \in k[x, y] \) of degree \( k \) to another curve defined by a polynomial of the same degree.