Evaluation, interpolation and multivariate multiplication

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joint work with Joris van der Hoeven (LIX, X)

The big picture

We have very good algorithms for univariate polynomials

- arithmetic operations $+, \times, \div$
- computing in $K \to K[X]/f$

We would like good algorithms for multivariate polynomials

- one question (among others):
  efficient arithmetic in $K \to K[X_1, \ldots, X_n]/\langle f_1, \ldots, f_s\rangle$
- application: solving polynomial systems

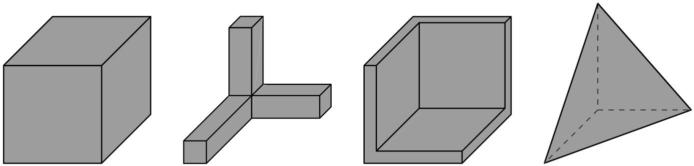
Objective: quasi-linear time, no factor of the form c^n in the cost
(in particular, expansion is forbidden – cf.
Canny-Kaltofen-Lakshman's sparse product)

This talk

No known solution, even for nice assumptions on $\langle f_1, \ldots, f_s \rangle$ such as

- tdeg or lex Gröbner basis
- triangular set

This talk: algorithms for multiplication modulo zero-dimensional monomial ideals, i.e. multiplication of power series.



Main result

Input

- M is a zero-dimensional monomial ideal in $K[X_1, \ldots, X_n]$
- $\delta_M = \dim_K K[X_1, \ldots, X_n]/M$
  this is the input and output size
- $\mathrm{reg}_M = \max \deg(m)$, for m a monomial not in M

Theorem

One multiplication modulo M can be done in $\tilde{O}(\delta_M \, \mathrm{reg}_M \, n)$ operations in K (provided K is large enough).

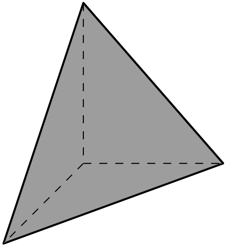
The factor reg_M is the price to pay to use evaluation and interpolation techniques.

Example: total degree truncation

Truncation in total degree is determined by

$$M = \langle X_1, \ldots, X_n \rangle^d = \langle \text{ all monomials of degree } d \,\rangle.$$

Used in many forms of Hensel lifting; the support is a simplex.

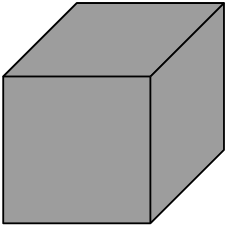


Example: partial degree truncation

Truncation in partial degree is determined by

$$M = \langle X_1^{d_1}, \ldots, X_n^{d_n} \rangle.$$

Used in a few (more marginal?) algorithms. The support of such series is a cube.



Previous work

1 variable

Truncation does not help for "optimal" algorithms (Fiduccia-Zalcstein)

Short product: improvement for algorithms like Karatsuba or Toom-Cook (Schönhage, Mulders, Hanrot-Zimmermann)

2 variables

Upper and lower bounds by Schönhage and Bläser

Total degree truncation

Quasi-linear cost for char $k = 0$ (Lecerf-S.)

Previous work by Griewank; refinments by van der Hoeven

Part I

Review: evaluation and interpolation in one variable

Polynomial multiplication

We let M be such that polynomials of degree less than n can be multiplied in $\mathsf{M}(n)$ base ring operations

Examples

- **Naive** $\mathsf{M}(n) = O(n^2)$
- **Karatsuba** $\mathsf{M}(n) = O(n^{\log_2(3)})$
- **Toom** $\mathsf{M}(n) = O(n^{\log_3(5)})$
- **FFT** over nice fields $\mathsf{M}(n) = O(n \log(n))$
- **FFT** in general $\mathsf{M}(n) = O(n \log(n) \log\log(n))$

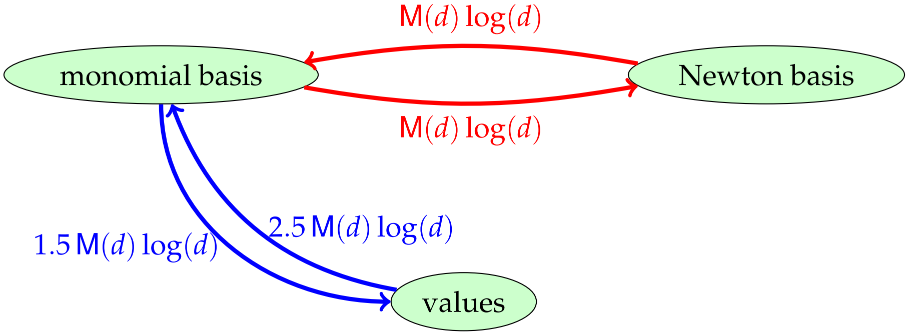
$+$ the assumptions of Chapter 9.

Evaluation and interpolation

Already in one variable, the problem comes in many different flavors:

- **polynomial** or rational
- **dense** or sparse
- **Lagrange**, Hermite, Birkhoff, …
- **monomial basis, Newton basis**, Bernstein basis, …
  - $1, X, \ldots, X^d$
  - $1, (X - x_0), \ldots, (X - x_0) \cdots (X - x_{d-1})$, x_i pairwise distinct

Known results (cf. Chapter 10)



Borodin-Moenck, Bostan-Lecerf-S.

Bini-Pan, Bostan-S.

Slightly better results for arithmetic progressions (Gerhard); much better results for geometric progressions (Bluestein, Rabiner *et al.*, Mersereau, Bostan-S.).

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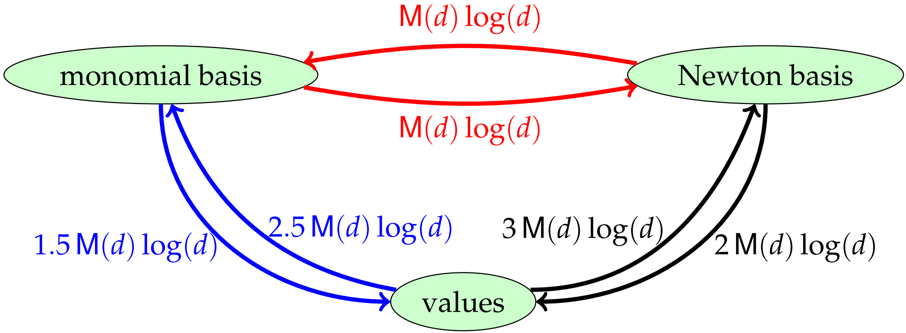

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Part II

Multivariate evaluation and interpolation

Previous work

In multivariate cases, there are many possible views on the problem (see Gasca-Sauer). From the point of view of feasibility, our interpolation problem will be simple.

Tensor product algorithms

- multidimensional FFT
- Pan (1994): simple evaluation / interpolation at a grid

Evaluation algorithms

- Nüsken, Ziegler (2004): bivariate evaluation at arbitrary points, subquadratic time
- Umans (2007), Kedlaya, Umans (2008): evaluation at arbitrary points, quasi-linear time

Setup

Choose an initial segment $T \subset \mathbb{N}^n$ for the partial order on $\mathbb{N}^n$

Monomial support

- $K[X_1, \ldots, X_n]_T = \{X_1^{i_1} \cdots X_n^{i_n} \mid (i_1, \ldots, i_n) \in T\}$
- bounding box: $d_1, \ldots, d_n$ such that

$$T \subset \{0, \ldots, d_1 - 1\} \times \cdots \times \{0, \ldots, d_n - 1\}$$

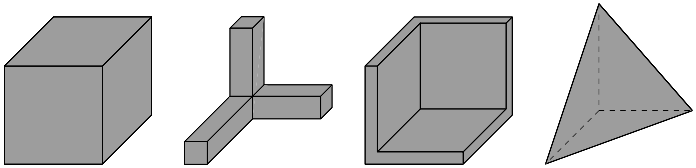
Sample set

- for $i \leq n$, pick pairwise distinct $x_{i,0}, \ldots, x_{i,d_i-1}$
- $V_T = \{ (x_{1,i_1}, \ldots, x_{n,i_n}) \mid (i_1, \ldots, i_n) \in T \}$
- V_T is contained in the grid

$$(x_{1,0}, \ldots, x_{1,d_1-1}) \times \cdots \times (x_{n,0}, \ldots, x_{1,d_n-1})$$

Examples

Monomial support



Easy sample set

- choose $(x_{i,0}, \ldots, x_{i,d_i-1}) = (0, \ldots, d_i - 1)$
- in this case $V_T = T$
- so we are evaluating polynomials supported on $K[X_1, \ldots, X_n]_T$ at the set T.

This is the sample set I will choose, even though using a geometric progression would be a bit better

Previous work

Mora, Sauer (but also Macaulay, Hartshorne . . .)

- the evaluation map is invertible
  (by a Gröbner basis argument)

Werner, 1980

- interpolation in the Newton basis, using divided differences

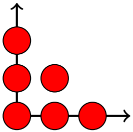
Multivariate Newton basis

Polynomials in $K[X_1, \ldots, X_n]_T$ can be written on:

- the monomial basis $X_1^{i_1} \cdots X_n^{i_n}$
- the Newton basis $N_{i_1}(X_1) \cdots N_{i_n}(X_n)$, with

$$N_{i_j}(X_j) = (X_j - x_{j,0}) \cdots (X_j - x_{j,i_j-1})$$

Example: T is given as



With a grid based on $(0,1,2) \times (0,1,2)$, the bases are

- $1,\ X_1,\ X_1^2,\ X_2,\ X_2X_1,\ X_2^2$
- $1,\ X_1,\ X_1(X_1 - 1),\ X_2,\ X_2X_1,\ X_2(X_2 - 1)$

Conversions

Theorem: Changes of basis can be done in time

$$O\left(\left(\frac{\mathsf{M}(d_1)\log(d_1)}{d_1}+\cdots+\frac{\mathsf{M}(d_n)\log(d_n)}{d_n}\right)|T|\right) \subset \tilde{O}(n|T|).$$

Done using a tensored version of the univariate algorithms.

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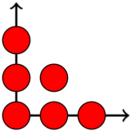

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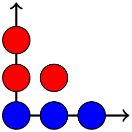

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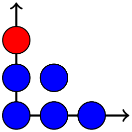

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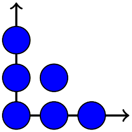

$\mathsf{M}(d_1)\log(d_1)\, d_2$

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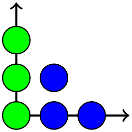

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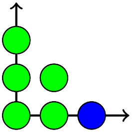

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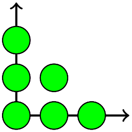

$$\mathsf{M}(d_1)\log(d_1)\,d_2\ +\ \mathsf{M}(d_2)\log(d_2)\,d_1$$

Evaluation and interpolation

Theorem: Evaluation and interpolation can be done in time

$$O\left(\left(\frac{\mathsf{M}(d_1)\log(d_1)}{d_1} + \cdots + \frac{\mathsf{M}(d_n)\log(d_n)}{d_n}\right)|T|\right) \subset \tilde{O}(n|T|).$$

Done in the Newton basis, using a tensored version of the univariate algorithms.

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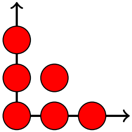

$$\begin{array}{lll} p_{0,2}X_2(X_2-1) & & \\ p_{0,1}X_2 & p_{1,1}X_1X_2 & \\ p_{0,0} & p_{1,0}X_1 & p_{2,0}X_1(X_1-1) \end{array}$$

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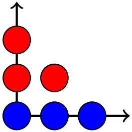

$p_{0,2}X_2(X_2-1)$

$p_{0,1}X_2 \qquad p_{1,1}X_1X_2$

$v_{0,0} \qquad v_{1,0} \qquad v_{2,0}$

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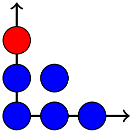

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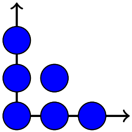

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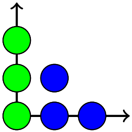

$w_{0,2}$

$w_{0,1}$ $\qquad$ $v_{1,1}X_2$

$w_{0,0}$ $\qquad$ $v_{1,0}$ $\qquad$ $v_{2,0}$

Evaluation and interpolation

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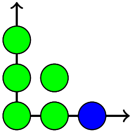

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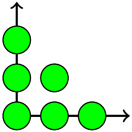

$w_{0,2}$		
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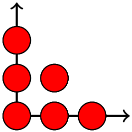

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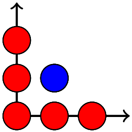

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To evaluate P at $(1,1)$, we just need the coefficients "under" $(1,1)$

Part III

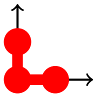
Power series multiplication

Review

Setup

- M: zero-dimensional monomial ideal in $K[X_1, \ldots, X_n]$
- T: exponents of the monomials not in M
- $\delta_M = \dim K[X_1, \ldots, X_n]/M = |T|$
  this is the input and output size
- $\mathrm{reg}_M = \max \deg(m)$, for m not in M

Example $M = \langle X_1^2,\ X_1X_2,\ X_2^2 \rangle$



- $T = \{(0,0),\ (1,0),\ (0,1)\}$
- $K[X_1, X_2]_T$ generated by $1, X_1, X_2$
- $\delta_M = 3$, $\mathrm{reg}_M = 1$

Evaluation / interpolation techniques

Theorem (2005). One multiplication modulo M can be done using

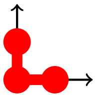
- $O(\mathrm{reg}_M)$ evaluations / interpolations at T of polynomials in $K[X_1, \ldots, X_n]_T$
- δ_M univariate power series products at precision $O(\mathrm{reg}_M)$

(at the time, I did not know how to do the evaluation / interpolation)

Ingredient: APA-algorithms (as in fast matrix multiplication)

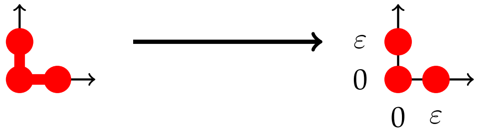
- Bini-Capovani-Romani-Lotti: floating-point products
- Bini: relation to exact computations
- Bini-Lotti-Romani, Schönhage: multiplication modulo X^2

The algorithm on an example



to multiply modulo
$\langle X_1^2,\ X_1X_2,\ X_2^2\rangle$

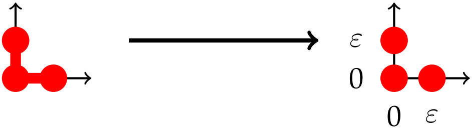
The algorithm on an example



to multiply modulo $\langle X_1^2,\ X_1X_2,\ X_2^2\rangle$

multiply modulo $\langle X_1(X_1-\varepsilon),\ X_1X_2,\ X_2(X_2-\varepsilon)\rangle$

The algorithm on an example



to multiply modulo
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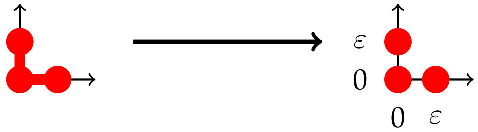
multiply modulo
$\langle X_1(X_1-\varepsilon),\ X_1X_2,\ X_2(X_2-\varepsilon)\rangle$

let $\varepsilon = 0$

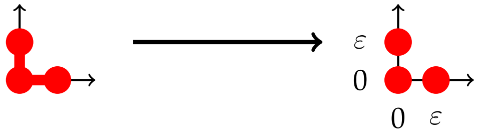


to multiply modulo
$\langle X_1^2,\ X_1X_2,\ X_2^2\rangle$

multiply modulo
$\langle X_1(X_1-\varepsilon),\ X_1X_2,\ X_2(X_2-\varepsilon)\rangle$
(by evaluation / interpolation)

let $\varepsilon = 0$

The algorithm on an example



to multiply modulo $\langle X_1^2,\ X_1X_2,\ X_2^2\rangle$

multiply modulo $\langle X_1(X_1-\varepsilon),\ X_1X_2,\ X_2(X_2-\varepsilon)\rangle$
(by evaluation / interpolation)

let $\varepsilon = 0$

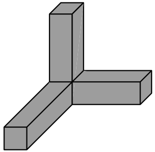
- in general:

  in every member $x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}$ of the basis of P change $x_i^{p_i}(i=1,\,2,\,\dots,\,n)$ to $x_i(x_i-1)\dots(x_i-p_i+1)$.

- do evaluation / interpolation, with power series coefficients
- correctness (again) from Mora-Sauer-…'s argument
- precision in ε = 2 × regularity

Going beyond

1. The factor reg_M is annoying
   - I should allow expansion: allows product modulo $\langle X_1^d, \ldots, X_n^d \rangle$ in time $\tilde{O}(\delta 4^{\sqrt{\log \delta}})$
   - but still, does not solve



2. Computing modulo a zero-dimensional Gröbner basis?
   - initial ideals obtained through one-parameter deformations
   - homotopy techniques?