1. Exercise sheet
Hand in solutions until Monday, 01 November 2010, 23:59h.

Reminders.

- For the course we remind you of the following dates:
  - Lectures: Tuesday and Wednesday 13:00h-14:30h sharp, b-it 1.25.
  - Tutorial: Tuesday 14:45h-16:15h sharp, Room 1.25.

- A word on the exercises. They are important. Of course, you know that. In order to be admitted to the exam it is necessary that you earned at least 20% of the credits. Just as an additional motivation, you will get a bonus for the final exam if you attended both lecture and tutorial regularly and earned more than 60% or even more than 80% of the credits.

Exercise 1.1 (Unitary transformations). (12 points)

A Hilbert-space $V$ is a complete complex vector space, equipped with a scalar product. We will always denote elements $x$ of $V$ by $|x\rangle$. In quantum information theory we only consider the Hilbert spaces $\mathbb{C}^n$ for some $n \in \mathbb{N}$ with the scalar product

$$\langle x | y \rangle := \sum_{i=1}^n \bar{x}_i y_i.$$ 

Define the length of $|x\rangle$ as $\sqrt{\langle x | x \rangle}$.

(i) Verify that the scalar product defined above has the following properties:

- Prove that the operation is linear in the second argument, i.e.
  $$\langle x | y_1 + y_2 \rangle = \langle x | y_1 \rangle + \langle x | y_2 \rangle$$
  and 
  $$\langle x | cy \rangle = c \langle x | y \rangle.$$ 

What can you say about the linearity in the first argument?

- $\langle x | y \rangle = \overline{\langle y | x \rangle}$,

- $\langle x | x \rangle \in \mathbb{R}_{\geq 0}$ with equality if and only if $|x\rangle = 0$.

For a matrix $U \in \mathbb{C}^{n \times n}$ define its adjoint as $(U^*)_i,j = \overline{U_j,i}$ i.e. as the the conjugated and transposed matrix of $U$. We call a matrix unitary if $U^{-1} = U^*$.

(ii) Show that for any matrix we have $\langle U^* x | y \rangle = \langle x | U y \rangle$. Hint: Express any vector in terms of the standars basis $e_1 = (1,0,\ldots,0)^T, \ldots, e_n = (0,\ldots,0,1)^T$ and write out both inner products.

(iii) Conclude that unitary matrices $U$ preserve the inner product, i.e. we have $\langle U x | U y \rangle = \langle x | y \rangle$. 


Exercise 1.2 (The toss of a fair coin). (8+5 points)

Consider a quantum system with two basic states “head” $|h\rangle$ and “tail” $|t\rangle$. We call this system a quantum coin. Consider a time evolution

$$
|h\rangle \mapsto \frac{1}{\sqrt{2}} |h\rangle + \frac{1}{\sqrt{2}} |t\rangle ,
$$

$$
|t\rangle \mapsto \frac{1}{\sqrt{2}} |h\rangle - \frac{1}{\sqrt{2}} |t\rangle ,
$$

which is called a fair coin toss.

(i) Write down the matrix representation of the fair coin toss and prove that it is unitary.

(ii) Verify that when we start with one of the states $|h\rangle$ or $|t\rangle$, after the toss, we will end up in a state where $|h\rangle$ and $|t\rangle$ are observed both with probability $\frac{1}{2}$.

(iii) Show that when we start with the state $|h\rangle$ and perform the coin toss twice, we end up again with state $|h\rangle$.

(iv) Interpret the result.