Advanced Cryptography: Quantum Cryptography

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2. Exercise sheet
Hand in solutions until Monday, 08 November 2010, 23:59h.

Exercise 2.1 (Matrix exponentials). (8 points)

For a matrix $H \in \mathbb{C}^{n\times n}$, define its exponential as

$$\exp(H) = \sum_{k\geq 0} \frac{1}{k!} \cdot H^k.$$

(i) Show that if H is Hermitian, which means $H = H^*$, then $\exp(-i \cdot Ht)$ is unitary. 3

(ii) Show that if H is diagonal, that is 2

$$H = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

then

$$\exp(H) = \begin{bmatrix} \exp(d_1) & 0 & \dots & 0 \\ 0 & \exp(d_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp(d_n) \end{bmatrix}.$$

(iii) Show that if H is diagonalizable, i.e. there is a matrix $T \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D \in \mathbb{C}^{n\times n}$ with $H = T^{-1}DT$ that then 3

$$\exp(H) = T^{-1}\exp(D)T.$$

Exercise 2.2 (Eigenvectors and spaces). (8 points)

Consider the real matrix

$$M = \begin{bmatrix} -1 & -4 & -4 \\ 1 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

(i) Verify that the vectors 3

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

are eigenvectors of M and compute the corresponding eigenvalues.

(ii) Rewrite M in terms of the basis v_1, v_2, v_3. 2

(iii) Draw conclusions of you observation. 3

Exercise 2.3 (Quantum mechanical postulates). (0+5 points)

+5 In the lecture we encountered four postulates of quantum theory. Show that postulate 2 is equivalent to postulate 2′.

Exercise 2.4 (Hamiltonian). (0+12 points)

Consider the (quantum) harmonic oscillator. The Hamiltonian of the system is given by

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2,$$

where this simple form uses natural units, i.e. energy in units of $\hbar\omega$ and disctance of units $\sqrt{\frac{\hbar}{m\omega}}$. The eigenvectors are $|\psi_n\rangle$ with

$$\psi_n(x) = \frac{1}{\sqrt{2^n \cdot n!}}\pi^{-1/4}\exp(-x^2/2)h_n(x),$$

using the n-th Hermite polynomial $h_n(x) = (-1)^n \cdot \exp(-x^2)\frac{\partial^n}{\partial x^n}\exp(-x^2)$.

+5 (i) Verify that the corresponding energy-levels (= eigenvalues) are $E_n = n + \frac{1}{2}$, i.e. verify that

$$H\,|\psi_n\rangle = E_n\,|\psi_n\rangle\,.$$

You may employ a computer algebra system to do so for $n \in \{0, 1, 2, 3, 4\}$.

+5 (ii) Now take any quantum state $|\psi\rangle = \sum_n \alpha_n(t)\,|\psi_n\rangle$ for some functions α. Use the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,|\psi\rangle = H\,|\psi\rangle$$

to get explicit expressions for α_n.

+2 (iii) What is the lowest possible energy? What is the classical energy of the ground state of a harmonic oscillator? Observations?