In the lecture we encountered a number of single-qubit gates. Important are the Hadamard gate $H$, the phase gate $S$ and the $\pi/8$-gate (denoted by $T$):

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, T = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix}.$$  

Also the Pauli gates $X$, $Y$ and $Z$ are of central importance:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

**Exercise 3.1 ($\pi/8$???)**. (1 points)

Find a reasonable explanation why the $T$ gate is called the $\pi/8$-gate.

**Exercise 3.2 (Rotation matrices)**. (14+5 points)

(i) Restate the eigenvalues of the Pauli matrices and find the points on the Bloch-sphere which correspond to the normalized eigenvectors.

(ii) The Pauli matrices give rise to three rotation operators:

$$R_x(\vartheta) = \exp(-i\vartheta X/2) = \cos \frac{\vartheta}{2} \cdot I - i \sin \frac{\vartheta}{2} \cdot X$$
$$R_y(\vartheta) = \exp(-i\vartheta Y/2) = \cos \frac{\vartheta}{2} \cdot I - i \sin \frac{\vartheta}{2} \cdot Y$$
$$R_z(\vartheta) = \exp(-i\vartheta Z/2) = \cos \frac{\vartheta}{2} \cdot I - i \sin \frac{\vartheta}{2} \cdot Z.$$  

Let $x \in \mathbb{R}$ and $A$ a matrix such the $A^2 = I$. Show that then

$$\exp(iAx) = \cos x \cdot I + i \sin x \cdot A$$

and, using this, verify the above three equations for the rotation operators. Hint: Remember $\cos x = \sum_{k=0}^{\infty} (-1)^k x^{2k} / (2k)!$ and $\sin x = \sum_{k=0}^{\infty} (-1)^k x^{2k+1} / (2k+1)!$.

(iii) Show that up to a global phase we have $T = R_z(\pi/4)$.

(iv) Express the Hadamard-gate $H$ as a product of $R_x(\vartheta_1), R_z(\vartheta_2)$ and $\exp(i\varphi)$ for some $\vartheta_1, \vartheta_2$ and $\varphi$.

(v) Prove in general that if $U$ is a unitary operation on a single qubit then there exist real numbers $\alpha, \beta, \gamma, \delta$ such that

$$U = \exp(i\alpha)R_z(\beta)R_y(\gamma)R_z(\delta).$$
Exercise 3.3 (Quantum teleportation). (11 points)

In the lecture we encountered a nice quantum circuit that performs quantum teleportation. We are now to prove that the following modified circuit also teleports a qubit:

\[ H \]

\[ X \]

\[ Z \]

(i) Write down the unitary transformation that describes the circuit. Hint: Proceed step by step. A computer algebra system might be of big help!

(ii) Prove that if \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \) then on input \( |\psi\rangle \otimes \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) the output of the circuit is \( H \otimes 2 |00\rangle \otimes |\psi\rangle \).

(iii) How can you now get rid of the first two qubits?

Exercise 3.4 (Experimental Quantum teleportation). (0+10 points)

Read and report on the article

http://www.nature.com/nature/journal/v390/n6660/full/390575a0.html