PRIME FLUX

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Abstract. This treatise summarizes various aspects related to prime counting. This includes pointers to several variants of the Prime Number Theorem (Theorem 1.2). Some aspects of the intimately related Riemann ζ-function are sketched and 'the' exact formula is described.
Further, we sketch several algorithms for counting primes exactly.
As side tracks, we started investigating primes in arithmetic progressions and other variations.

Keywords. Number theory, prime counting, Riemann ζ, algorithms.

Subject classification. ??

Part I

Global estimates

1. Counting primes

Consider the number $\pi(x)$ of primes less or equal to x, or as we prefer it:

$$\pi(x) = \sum_{p \le x}^{\mathbb{P}} 1. \tag{1.1}$$

Here the sum runs over all primes(!) p less or equal than x. Whenever a sum runs over a parameter p or p' or p_i or q it shall be self understood that it is a sum over primes only. Today, various estimates for this count are available:

THEOREM 1.2 (Prime number theorem). *We list a few variants:*

(i) *Chebyshev (1852), conjectured by Legendre (1798):*

$$\pi(x) \approx \frac{x}{\ln x}.$$

(ii) *Hadamard (1896), de la Vallée Poussin (1896), Walfisz (1963), conjectured by Gauß (1849):*

$$\begin{aligned}\pi(x) &\in \mathrm{Li}(x) + \mathcal{O}\left(x \exp\left(-\frac{A(\ln x)^{3/5}}{(\ln\ln x)^{1/5}}\right)\right)\\ &\subset \mathrm{Li}(x) + \mathcal{O}\left(\frac{x}{\ln^k x}\right)\end{aligned}$$

for any k. The presently best known value for A is $A = 0.2098$ (Ford 2002a, p. 566).[1]

Here, the logarithmic integral Li *is given by* $\mathrm{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\ln t}$. *Other authors use* $\mathrm{li}(x) = \int_0^x \frac{\mathrm{d}t}{\ln t}$, *which differs from* $\mathrm{Li}(x)$ *by the additive constant* $\int_0^2 \frac{\mathrm{d}t}{\ln t} = 1.045⊥$.[2]

(iii) *Dusart (1998, Théorème 1.10, p. 36): For $x \ge 355991$ we have*

$$\frac{x}{\ln x} + \frac{x}{\ln^2 x} + 1.8\frac{x}{\ln^3 x} < \pi(x) < \frac{x}{\ln x} + \frac{x}{\ln^2 x} + 2.51\frac{x}{\ln^3 x}.$$

This includes the classical estimate $\pi(x) < \frac{2x}{\ln x}$.

(iv) *Von Koch (1901), Schoenfeld (1976): If (and only if) the Riemann hypothesis holds then for $x \ge 2657$*

$$|\pi(x) - \mathrm{li}(x)| < \frac{1}{8\pi}\sqrt{x}\ln x.$$

If (and only if) the Riemann hypothesis holds then for $x \ge 1451$

$$|\pi(x) - \mathrm{Li}(x)| < \frac{1}{8\pi}\sqrt{x}\ln x.$$

(v) *Cramér (1935, 1937): If primes were random with* $\mathrm{prob}(n \text{ prime}) = \frac{1}{\ln n}$ *then their counting function* $\Pi(x)$ *fulfills*

$$|\Pi(x) - \mathrm{Li}(x)| < \sqrt{\frac{2x\ln\ln x}{\ln x}}$$

asymptotically surely, and even more:

$$\mathrm{prob}\left(\limsup_{x\to\infty} \frac{|\Pi(x) - \mathrm{Li}(x)|}{\sqrt{2\ln\ln x}\sqrt{\frac{x}{\ln x}}} = 1\right) = 1.$$

The prime counting function π gets three companions: the Riemann prime counting function π^* (also denoted Π, π_*, J, or as by Riemann f), the Chebyshev functions ϑ and ϑ^* (also denoted ψ).

$$\pi(x) = \sum_{p\le x} 1, \qquad \pi^*(x) = \sum_{p^k \le x} \frac{1}{k} \quad = \sum_{n \le x} \frac{\Lambda(n)}{\ln n}, \tag{1.3}$$

$$\vartheta(x) = \sum_{p\le x} \ln p, \qquad \vartheta^*(x) = \sum_{p^k \le x} \ln p \quad = \sum_{n\le x} \Lambda(n). \tag{1.4}$$

Here the von Mangoldt function Λ is defined by

(1.5)

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^k \text{ for some } p \in \mathbb{P} \text{ and some } k \in \mathbb{N}_{>0},\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\frac{\Lambda(n)}{\ln n} = \frac{1}{k}$ for a prime power $n = p^k$. To formulate the simple relation between some of these we need the Möbius function

$$\mu(n) = \begin{cases} +1 & \text{if } n \text{ is a product of an \emph{even} number of distinct primes,}\\ -1 & \text{if } n \text{ is a product of an \emph{odd} number of distinct primes,}\\ 0 & \text{otherwise, that is, if } n \text{ is squareful.}\end{cases} \tag{1.6}$$

For $n > 1$ we have $\sum_{d|n} \mu(d) = 0$ whereas $\sum_{d|1} \mu(d) = 1$. If now $g(n) = \sum_{d|n} f(d)$ then $f(n) = \sum_{d|n} \mu(d) g(n/d)$.[i]

The prime counting functions are related pairwise by the equations

$$\pi(x) = \sum_{n\ge 1} \frac{\mu(n)}{n} \pi^*(x^{\frac{1}{n}}), \qquad \pi^*(x) = \sum_{n \ge 1} \frac{1}{n} \pi(x^{\frac{1}{n}}), \tag{1.7}$$

$$\vartheta(x) = \sum_{n\ge 1} \mu(n) \vartheta^*(x^{\frac{1}{n}}), \qquad \vartheta^*(x) = \sum_{n\ge 1} \vartheta(x^{\frac{1}{n}}). \tag{1.8}$$

Furthermore, for each of these we add a variant with an mean value at the jumps. For instance, $\pi_0(x) = \lim_{\varepsilon\to 0} \frac{\pi(x-\varepsilon)+\pi(x+\varepsilon)}{2}$. Actually, this is related to rewriting expressions as Riemann-Stieltjes integrals, which are then further transformed. For example,

$$\sum_{p\le x}^{\mathbb{P}} f(p) = \int_1^x f(y)\ \mathrm{d}\pi(y) \tag{1.9}$$

To give a further indication of their relation let me tell you about the work of Rosser & Schoenfeld (1962, 1975) and

[1] The only estimate that we found with an explicitly given 'constant' has $A = \frac{1}{57}$ and $a = 11.88\ln^{3/5} x$ (well, that's not constant, but almost...), Ford (2002b) attributes this to Y. Cheng, *Explicit estimates on prime numbers (pre-print)*. [Though there are several preprints on arXiV (http://arxiv.org/abs/0810.2113v1, http://arxiv.org/abs/0810.2102v3, http://arxiv.org/abs/0810.2103v5), we couldn't find that paper...]

[2] Side remark: to indicate how a real number was rounded we append a special symbol. Examples: $\pi = 3.14$⊥ $= 3.142$Ұ $= 3.1416$Ƭ $= 3.14159$⊥. The height of the platform shows the size of the left-out part and the direction of the antenna indicates whether actual value is larger or smaller than displayed. We write, say, $e = 2.72$Ƭ $= 2.71$⊓ as if the shorthand were exact.

Schoenfeld (1976). Actually, Theorem 1.2(iv) follows from Schoenfeld's following statement about the Chebyshev function ϑ^*: for $x \geq 73.2$

$$|\vartheta^*(x) - x| < \frac{1}{8\pi}\sqrt{x}\ln^2 x.$$

Following roughly Schoenfeld (1976) we derive Theorem 1.2(iv) as follows. Noting $\mathrm{d}\pi(y) = \frac{\mathrm{d}\vartheta(y)}{\ln y}$ we can reformulate $\pi(x)$ and then integrate by parts:

$$\begin{aligned}\pi(x) &= \int_{2-}^{x} \frac{\mathrm{d}\vartheta(t)}{\ln t} \\ &= \frac{\vartheta(x)}{\ln x} + \int_{2-}^{x} \frac{\vartheta(t)}{t\ln^2 t}\,\mathrm{d}t\,.\end{aligned}$$

Performing the same for $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\ln t}$ and subtracting it gives

$$\pi(x) - \mathrm{Li}(x) = \frac{\vartheta(x) - x}{\ln x} + \int_{2-}^{x} \frac{\vartheta(t) - t}{t\ln^2 t}\,\mathrm{d}t\,.$$

By the bound for ϑ^* we can bound the first summand with the desired $\frac{1}{8\pi}\sqrt{x}\ln x$. It is obvious that the integral term is negligible and a better analysis shows that it does not even influence the constant.

2. Riemann's ζ-function

No version of Theorem 1.2 was provable before Riemann (1859) found the fundamental connection between primes and the now so-called Riemann ζ-function:

$$\begin{aligned}
&(2.1) & \zeta(s) &= \sum_{n\geq 1}\frac{1}{n^s} && \text{for } \Re s > 1\\
&(2.2) & &= \prod_{p}\frac{1}{1-p^{-s}} && \text{for } \Re s > 1\\
&(2.3) & &= \frac{1}{1-2^{1-s}}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^s} && \text{for } \Re s > 0.\\
&(2.4) & &= \frac{\Gamma(1-s)}{2\pi i}\int_P \frac{\exp(s\ln(-z))}{\exp(z)-1}\,\mathrm{d}z && \text{for } s \neq 1,
\end{aligned}$$

where P denotes a path 'once around the positive real axis', that is, it starts at $\infty + i\delta$ travels to $0 + i\delta$ then circles around 0 and finally travels from $0 - i\delta$ back to $\infty - i\delta$ for some small δ (which excludes any poles which are not on the axis). Next, the ζ function fulfills the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{2.5}$$

Riemann (1859) introduced the function

$$\begin{aligned}\xi(s) &:= (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s) \\ &= \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\,,\end{aligned} \tag{2.6}$$

which is defined for all $s \in \mathbb{C}$ and has no poles. Now, the functional equation can be expressed as

$$\xi(s) = \xi(1-s). \tag{2.7}$$

Due to the functional equation ζ can be extended to the entire complex plane with the only, single pole at $s = 1$. The zeroes of the sine function at negative integers (the non-negative zeroes are cancelled with poles of Γ, the zero at $s = 0$ with the pole of ζ), or in the ξ form the single poles of the Γ function at non-positive integers, imply that ζ has *trival* zeros at even negative integers $-2\mathbb{N}_{>0}$. In other words, ξ has no pole and only zeroes with $\mathrm{Re}(s) \in [0, 1]$. Moreover, $\xi(\frac{1}{2} + it)$ is real for $t \in \mathbb{R}$.

One of the two first proofs[3] of the prime number theorem Theorem 1.2 is based on the fact that ζ has no zeroes on the line where $\Re s = 1$, which Hadamard (1896) proved by an ingenious trick. He observed that for $x > 1$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}&|\zeta(x)^3\zeta(x+it)^4\zeta(x+2it)| \\ &\quad = \exp\left(\sum_{n\geq 2}\frac{2^{\frac{\Lambda(n)}{\ln n}}}{n^x}(1+\cos(t\ln n))^2\right) \geq 1\end{aligned} \tag{2.8}$$

has absolute value at least 1 for all $x > 1$. (Note that $2(1 + \cos(t\ln n))^2 = 3 + 4\cos(t\ln n) + \cos(2t\ln n)$, explaining the clever choice of the exponents $3, 4, 1$.) If now $\zeta(1 + it)$ were zero then the mentioned product would also have a zero there (the triple pole would meet an at least four-fold zero). But taking the limit $x \to 1$ then leads to a contradiction.

This simple fact implies the prime number theorem with an error bound of order

$$\mathcal{O}(x\exp(-c\sqrt{\ln x}))$$

on the error $\pi(x) - \mathrm{Li}(x)$. Since then the prime number theorem has been improved repeatedly by establishing larger zero-free regions within the critical strip $[0, 1] + i\mathbb{R}$.

3. Relations

Riemann (1859) found an explicit formula for π using the zeroes of the ζ-function:

$$\pi_0^*(x) = \mathrm{Li}(x) - \sum_{\varrho}\mathrm{Li}(x^\varrho) + \int_x^\infty \frac{\mathrm{d}t}{t(t^2-1)\ln t} - \ln 2, \tag{3.1}$$

where the sum runs over all zeroes of ζ with real part between zero and one.[4] A more elegant variant of (3.1) is by von Mangoldt (1895)

$$\vartheta_0^*(x) = x - \sum_{\varrho}\frac{x^\varrho}{\varrho} - \underbrace{\frac{1}{2}\ln\left(1-\frac{1}{x^2}\right)}_{=\sum_{n\geq 1}\frac{x^{-2n}}{-2n}} - \underbrace{\ln(2\pi)}_{=\frac{\zeta'(0)}{\zeta(0)}}. \tag{3.2}$$

[3] The other is by de la Vallée Poussin (1896).

[4] Actually, this is only correct if $\mathrm{Li}(x^\varrho)$ is considered an imprecise shortcut for $\mathrm{Ei}(\varrho\ln x)$ where $\mathrm{Ei}(z) = \gamma + \ln(z) + \mathrm{Ei}_0(z)$, $\mathrm{Ei}_0(z) = \sum_{k\geq 1}\frac{z^k}{k!\cdot k} = \int_0^z \frac{\exp(t)-1}{t}\,\mathrm{d}t = \int_0^1 \frac{\exp(uz)-1}{u}\,\mathrm{d}u$, is the exponential integral.[ii] The problem is how the analytic continuation is done. For $z = x^\varrho$ on the way from $x^{\Re(\varrho)}$ til x^ϱ we would run $\frac{1}{2\pi}\Im(\varrho)$ times around zero, a branch point of the logarithmic integral Li. However, in the form $\mathrm{Ei}(\varrho\ln x)$ we just run in a strait line from $\Re\varrho\cdot\ln x$ to $\varrho\ln x$ without circling around a branch point.

To obtain a more explicit expression for the prime counting function π itself, we can use its relation to π^* and obtain:

$$\pi_0(x) = R(x) - \sum_{\varrho} R(x^{\varrho}) - \frac{1}{\ln x} + \frac{1}{\pi}\arctan\frac{\pi}{\ln x}, \tag{3.3}$$

where

$$R(x) = \sum_{n\geq 1}\frac{\mu(n)}{n}\,\mathrm{Li}(x^{\frac{1}{n}}) = 1 + \sum_{k\geq 1}\frac{(\ln x)^k}{k!k\zeta(k+1)}. \tag{3.4}$$

The second description of R converges fast. Kulsha (2008) gives a proof of (3.3) on the basis of (3.1). The problematic part is the conditional convergence of the sum over the non-trivial zeros.[iii]

Proofs of Riemann's formula (3.1) can be found in many textbooks on analytic number theory. However, most modern presentations prove the more elegant version (3.2) of von Mangoldt. A classical textbook covering both is Edwards (1974). One quite precise account can be found in Kunik (2005), who also derives (3.1) from (3.2). The only available proof for (3.3) seems to be by Kulsha (2008), which is completely based on known formulas from several sources. These proofs often use quite precise bounds, which were stated by Riemann (1859) and proved by von Mangoldt (1905), on the number $N(T)$ of zeroes of the ζ-function within a bounded region $[0,1] + i[0,T]$:

$$N(T) = \frac{T}{2\pi}\ln\frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\ln T). \tag{3.5}$$

Let us at least give an idea of the connection between zeros of ζ and the number of primes. We start with the Euler product representation of ζ and rewrite this in the form[5]

$$\ln\zeta(s) = \sum_{p}^{\mathbb{P}}\sum_{k\geq 1}\frac{1}{k}p^{-ks}. \tag{3.6}$$

This is a connection between all of ζ and all primes. We still need to 'select' the primes below some bound x. This is achieved using the Mellin transform of the Heaviside function $\varphi(u) = H(1-u)$ flipped about $\frac{1}{2}$. The Mellin transform maps a function f defined at least on $\mathbb{R}_{>0}$ with $f(x) = \frac{f(x-)+f(x+)}{2}$ to a function $\mathcal{M}f$ by $(\mathcal{M}f)(s) = \int_0^\infty f(u)u^s\,\frac{\mathrm{d}u}{u}$. Like the Fourier transform, which is a close relative[6], the Mellin transform has an inverse: $f(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}(\mathcal{M}f)(s)u^{-s}\,\mathrm{d}s$. For the named step function φ given by

$$\varphi(u) = \begin{cases} 1 & \text{if } u < 1,\\ \frac{1}{2} & \text{if } u = 1,\\ 0 & \text{if } u > 1,\end{cases} \tag{3.7}$$

we obtain its Mellin transfrom

$$\begin{aligned}\Phi(s) &= (\mathcal{M}\varphi)(s)\\ &= \int_0^1 u^{s-1}\,\mathrm{d}u = \frac{1}{s}.\end{aligned} \tag{3.8}$$

Here we obtain

$$\varphi(u) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(s)u^{-s}\,\mathrm{d}s \tag{3.9}$$

provided a is not too small, that is, $a > 1$ in our case (Edwards 1974, §3.3). Thus the Mellin transform 'selects' all terms with $u \leq x$. Applying the inverse transform to $\frac{x^s}{s}$ times (3.6) yields:

$$\begin{aligned}&\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s}\ln\zeta(s)\,\mathrm{d}s\\ &= \sum_{p}^{\mathbb{P}}\sum_{k}\frac{1}{k}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{s}\left(\frac{p^k}{x}\right)^{-s}\mathrm{d}s\\ &= \sum_{p}^{\mathbb{P}}\sum_{k}\frac{1}{k}H(1-x/p^k) = \sum_{p^k\leq x}^{\mathbb{P}}\frac{1}{k}.\end{aligned}$$

Read backwards we have Riemann's formula

$$\pi^*(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s}\ln\zeta(s)\,\mathrm{d}s\,. \tag{3.10}$$

It is now obvious to a complex analyst that the zeroes of ζ must play a decisive rôle as these are the values where the integrand has poles.

Von Mangoldt proposes another variant (Edwards 1974, §3.2). Instead of the Euler product formula itself, he starts with its logarithmic derivative, that is, compute the derivative of (3.6):

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 2}\Lambda(n)n^{-s} = \int_0^\infty x^{-s}\,\mathrm{d}\vartheta^*(x)\,. \tag{3.11}$$

By suitable inverse transformation this leads to

$$\vartheta^*(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} -\frac{\zeta'(s)}{\zeta(s)}x^s\,\frac{\mathrm{d}s}{s}. \tag{3.12}$$

The integral can be reevaluated based on the zeros of its integrand[7] which ends up in (3.2).

4. Fluctuations

Many definitions for displaying the fluctuations are around. Gourdon & Sebah (2001) propose

$$\theta(x) = \frac{\pi(x) - \mathrm{Li}(x)}{\sqrt{\frac{2x\ln\ln x}{\ln x}}} \tag{4.1}$$

obviously based on Theorem 1.2(v). We have analyzed

$$\Delta_{\text{Schoenfeld}}(x) = \frac{\pi(x) - \mathrm{Li}(x)}{\frac{1}{8\pi}\sqrt{x}\ln x} \tag{4.2}$$

based on Theorem 1.2(iv) and verified that $|\Delta_{\text{Schoenfeld}}(x)| \leq 1$ for $x \leq 2^{40}$. Kulsha (2008) considers

$$\Delta(x) = \frac{\pi_0(x) - \left(R(x) - \frac{1}{\ln x} + \frac{1}{\pi}\arctan\frac{\pi}{\ln x}\right)}{\frac{\sqrt{x}}{\ln x}} \tag{4.3}$$

[5] One needs to spend some extra care on the details if s is not real, but it comes out as presented here.

[6] The Mellin transform of f is a rotated variant of the Fourier transform of the function $f \circ \exp$: $(\mathcal{M}f)(is) = \sqrt{2\pi}(\mathcal{F}(f\circ\exp))(s)$. This can be seen using a change of variables $u = e^t$: $(\mathcal{M}f)(is) = \int_{-\infty}^{\infty} f(e^t)e^{ist}\,\mathrm{d}t$. However, usually that connection is not used.

[7]
$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} - \sum_{\varrho}\frac{s}{\varrho(s-\varrho)} - \sum_{k\geq 1}\frac{s}{-2k(s+2k)} - \frac{\zeta'(0)}{\zeta(0)}.$$

based on (3.3) and, probably, the observation that the second largest term in the definition is $\frac{1}{2}\operatorname{Li}(x)$ is asymptotically equal to $\frac{\sqrt{x}}{\ln x}$. Notice that we only have $\Delta(x) \in \mathcal{O}\left(\ln^2 x\right)$ under the Riemann hypothesis by Theorem 1.2(iv). Even in the model by Cramér (1935, 1937) we only expect $|\Delta(x)| \lesssim \sqrt{\frac{\ln x}{2\ln\ln x}}$. Nevertheless, plots for x up to 2^{76} seem to indicate that $|\Delta(x)| \leq 1$, which is definitely wrong in the long run.

Despite the contemporal believe that $\pi(x) < \operatorname{Li}(x)$, Littlewood (1914) observed that $\pi(x) - \operatorname{Li}(x)$ changes sign infinitely often and even exceeds $\Omega_\pm\left(\frac{\sqrt{x}}{\ln x}\ln\ln\ln x\right)$. Skewes (1933) proved that the first sign change occurs before $\exp(\exp(\exp(79))) < 10^{10^{10^{34}}}$ (under Riemann), a bound which has been improved to $1.39822 \cdot 10^{316}$ by Bays & Hudson (2000). Based on extensive calculations using up to $2 \cdot 10^{10}$ critical zeroes, Demichel (2005) suggests that the first crossover point is near the slightly smaller value $1.397162914 \cdot 10^{316}$. Kotnik (2008) claims based on computer calculations that the first crossover point is beyond 10^{14}.

Moreover, Wintner (1941) showed that the density of x-values with $\pi(x) > \operatorname{Li}(x)$ is positive, and Rubinstein & Sarnak (1994) proved that this proportion is at least $2.6 \cdot 10^{-7}$.

By definition $R(x)$ is about $\frac{1}{2}\operatorname{Li}(\sqrt{x}) \sim \frac{\sqrt{x}}{\ln x}$ larger than $\operatorname{Li}(x)$ and or 1 within $\Delta(x)$. Thus $\Delta(x)$ must become larger than 1 infinitely often.

4.1. A remark on beauty and truth. Kotnik (2008) discusses whether $\operatorname{Li}(x)$ or $r(x) := R(x) - \frac{1}{\ln x} + \frac{1}{\pi}\arctan\frac{\pi}{\ln x}$ is a better approximation for $\pi(x)$. The problem is that Li and r are much closer together than the maximal fluctuations of π. A way out could be to consider the 'average' error $\int_2^x (\pi(u) - r(u))\,\mathrm{d}u$ versus $\int_2^x (\pi(u) - \operatorname{Li}(u))\,\mathrm{d}u$. By the results of Littlewood (1914) I guess that again both terms oscillate much more than the difference and so probably no statement can be made. A variant would use relative errors instead of absolute errors (so divide both approximately by $\frac{x}{\ln x}$). Unfortunately, then the difference between r and Li tends to zero. So the absolute error sees too much and the relative error too few. Rescaling with $\frac{\sqrt{x}}{\ln x}$, which is the order of the difference $\operatorname{Li}(x) - r(x) \approx \frac{1}{2}\operatorname{Li}(x^{\frac{1}{2}})$. So consider $\Delta(x)$ in favour of r and $\Xi(x) := \frac{\pi_0(x) - \operatorname{Li}(x)}{\frac{\sqrt{x}}{\ln x}}$ in favour of Li. Asymptotically, $\Xi(x) \sim \Delta(x) + 1$. This difference is not much in the light of Littlewood (1914) implying both exceeding $\ln\ln\ln x$ in both directions infinitely often. But the average value of these quantities might be more informative. So consider

$$Q(\pi - f)(x) = \frac{\int_2^x \frac{\pi(u) - f(u)}{\frac{\sqrt{u}}{\ln u}} \frac{\mathrm{d}u}{u}}{\int_2^x \frac{\mathrm{d}u}{u}} \tag{4.4}$$

as a measure of quality. Notice that we use a logarithmic mean here as we expect the fluctuations to oscillate only logarithmically, that is, like $x \mapsto \sin(u \ln x)$. My guess is that r is not only the more beautiful approximation for π, but this will quantify how r approximates π better than Li.

As the formula work is much easier in von Mangoldt's variant, consider (3.2) and let $t(x) := x - \frac{1}{2}\ln\left(1 - \frac{1}{x^2}\right) - \ln(2\pi)$. Now, we try to get hold of $q_{\vartheta^* - t}$ by considering the summands in the remaining sum of the explicit formula (3.2). We have to adapt Q by removing the $\ln u$ factor from the denomiator, call this Q^*. Actually, one can check that, assuming $\operatorname{Re}\varrho = \frac{1}{2}$,

$$Q^*\left(u \mapsto \frac{u^\varrho}{\varrho} + \frac{u^{\overline{\varrho}}}{\overline{\varrho}}\right)(x) \in \mathcal{O}\left(\frac{1}{\varrho\overline{\varrho}\ln x}\right), \tag{4.5}$$

whereas the difference of Li and r translated here amounts to something like

$$Q^*\left(u \mapsto \sqrt{u}\right)(x) = 1 \tag{4.6}$$

Thus the influence of the difference should be visible in Q^*, and thus also the difference of r and Li in Q:

CONJECTURE 4.7. *The average quality* $|Q(\pi - r)|$ *is asymptotically smaller than* $|Q(\pi - \operatorname{Li})|$. *In this sense, Riemann's formula provides a better approximation than the logarithmic integral* Li.

Orthogonal to this approach, one might consider the sign distribution of $\pi - \operatorname{Li}$. As mentioned above, Rubinstein & Sarnak (1994) prove that the positivity domain of $\pi - \operatorname{Li}$ has logarithmic density of about $2.6 \cdot 10^{-7}$. It seems definite that it is much smaller than $\frac{1}{2}$. The sign density for $\pi - r$, however, seems to be uninvestigated. It thus seems natural to state the

CONJECTURE 4.8. *The logarithmic density of the positivity domain of* $\pi(x) - \operatorname{Li}(x)$ *is less than* 10^{-6}. *However, the logarithmic density of the positivity domain equals* $\frac{1}{2}$ *for each of the three functions*

- $\pi(x) - r(x)$,
- $\pi^*(x) - \left(\operatorname{Li}(x) + \int_x^\infty \frac{\mathrm{d}t}{t(t^2-1)\ln t} - \ln 2\right)$,
- $\vartheta^*(x) - \left(x - \frac{1}{2}\ln\left(1 - \frac{1}{x^2}\right) - \ln(2\pi)\right)$.

Mind that this conjecture is based solely on beauty and a some tiny evidence.

Part II
Progressed estimates

5. Counting primes in arithmetic progressions

Let q be any number and a coprime to q. Consider the number $\pi_{a+q\mathbb{Z}}(x) = \pi(x, q, a)$ of primes less than or equal to x that are congruent to a modulo q, or

$$\pi_{a+q\mathbb{Z}}(x) = \sum_{\substack{p \le x \\ p \in a+q\mathbb{Z}}}^{\mathbb{P}} 1. \tag{5.1}$$

Correspondingly, we define $\pi^*_{a+q\mathbb{Z}}(x)$, $\vartheta_{q+a\mathbb{Z}}(x)$, $\vartheta^*_{q+a\mathbb{Z}}(x)$. Here, we are collecting estimates to these quantities.

THEOREM 5.2 (Rubinstein & Sarnak 1994). *We have*

$$\begin{aligned}\varphi(q)\pi(x, q, a) = \pi(x) &- \left(\frac{a}{q}\right)\frac{\sqrt{x}}{\ln x} \\ &+ \frac{1}{\ln x}\sum_{\chi \ne \chi_0} \overline{\chi}(a)\vartheta^*_\chi(x) \\ &+ \mathcal{O}\left(\frac{\sqrt{x}}{\ln^2 x}\right),\end{aligned}$$

where $\vartheta^*_\chi(x) = \sum\limits_{n \le x} \chi(n)\Lambda(n)$ *and* χ_0 *is the principal character.*

Recall that a Dirichlet character χ is a function $\chi\colon \mathbb{N} \to \mathbb{C}$ which is multiplicative, periodic, and has $\chi(n) \ne 0$ iff n is coprime to the period. Every Dirichlet character of period k induces a character of $\mathbb{Z}_k^\times$ and vice versa. This is why the period is usually called modulus. The smallest period of a character is called its conductor. A character is primitive for a modulus k if the character has no smaller periiod. The unique character of period 1 is the principal character χ_0 and is given by $\chi_0(n) = 1$ for all n.[iv]

★¿★"Note that the characters of modulus k form a group and linearly span all k-periodic functions $\mathbb{N} \to \mathbb{C}$." is wrong! Find the correct statement!★?★ We thus consider the Dirichlet L-functions given by

$$L(s, \chi) = \sum_{n \ge 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1} \tag{5.3}$$

for $\Re s > 1$. Clearly, $L(s, \chi_0) = \zeta(s)$. By the definition it is clear that $L(s, \chi)$ is analytic for $\Re s > 1$, and by the Euler product description it is evident that $L(s, \chi)$ has no zeroes for $\Re s > 1$.

Consider a primitive character χ of modulus k and define $a \in \{0, 1\}$ by $\chi(-1) = (-1)^a$. Analoguously to ξ for ζ, consider

$$\Lambda(s, \chi) = \left(\frac{\pi}{k}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi). \tag{5.4}$$

Note that $\xi(s) = \frac{s(s-1)}{2}\Lambda(s, \chi_0)$. Corresponding to the functional equation of the ζ-function one finds the functional equation

$$\Lambda(1 - s, \overline{\chi}) = \frac{i^a\,|\tau(\chi)|}{\tau(\chi)}\Lambda(s, \chi). \tag{5.5}$$

The Gauß sum $\tau(\chi) = \sum_{n \in \mathbb{N}_{<k}} \exp\left(\frac{2\pi i n}{k}\right)\chi(n)$ has absolute value $\sqrt{k}$. This functional equation proves that $\Lambda(s, \chi)$ has no zeroes (nor poles) outside the strip $[0, 1] \times i\mathbb{R}$.

GENERALIZED RIEMANN HYPOTHESIS. *For any Dirichlet character χ all zeroes of $L(s, \chi)$ in the strip $[0, 1] + i\mathbb{R}$ have real part $\frac{1}{2}$.*

The special case $\chi = \chi_0$ is the Riemann hypothesis.

★¿★Weil's explicit formula★?★

Part III
Variations

6. Zeta functions for P-smooth numbers (?)

Bernier (2011) considers partial zeta functions. We rephrase his questions in our language. Let $\mathcal{A}$ be any subset of $\mathbb{R}$. A number n is called $\mathcal{A}$-grained if all its prime factors are in $\mathcal{A}$. Of course, we can always choose $\mathcal{A} \subset \mathbb{P}$, but we like the additional freedom. Now define the zeta function of $\mathcal{A}$.

$$\zeta_{\mathcal{A}}(s) := \sum_{n\ \mathcal{A}\text{-grained}} \frac{1}{n^s}.$$

This converges absolutely for $\mathrm{Re}(s) > 1$ since $\sum_{n>1} n^{-s}$ does for $s > 1$. Bernier asks whether there is an Euler product formula

$$\zeta_{\mathcal{A}}(s) = \prod_{p \in \mathcal{A} \cap \mathbb{P}} (1 - p^{-s})^{-1}.$$

This answer is easy: Yes, we have this for $\mathrm{Re}(s) > 1$. Moreover, depending on the set $\mathcal{A}$ it may well be that both expressions converge also for other values of s. Bernier requires $\mathcal{A} \cap \mathbb{P}$ to be infinite, but if we drop that both expressions are defined all over $\mathbb{C}$.

QUESTION 6.1. *Is there a functional equation fo $\zeta_{\mathcal{A}}$?*

Bernier motivates this as follows: Consider the number of primes $\pi_{\mathcal{A}}(x)$ in $\mathcal{A}$ less than or equal to x. Now, he claims that by picking $\mathcal{A}$ suitably one can achieve (under RH) that

$$|\pi_{\mathcal{A}}(x) - \mathrm{Li}(x)| \leq \begin{cases} \frac{1}{8\pi} \frac{\sqrt{x}}{\ln x} & \text{for } x < C, \\ x^{0.51+\varepsilon} & \text{otherwise,} \end{cases}$$

but there are arbitrarily large values x (beyond C, of course) such that

$$|\pi_{\mathcal{A}}(x) - \mathrm{Li}(x)| > x^{0.51}.$$

To that end he describes a set $\mathcal{A}$ which is all primes minus a very sparse set of primes. Namely, he picks C as Graham's number (which is just some very, very large integer for our purpose), $c_0 = \mathrm{nextprime}(C)$, $c_k := \mathrm{nextprime}(2^{c_{k-1}})$ and considers $\mathcal{A} = \mathbb{P} \setminus \{c_i \mid i \in \mathbb{N}\}$.

My feeling is that Li is just the wrong function here and should be replaced by a suitable variant $\mathrm{Li}_{\mathcal{A}}$ or Riemann's version $R_{\mathcal{A}}$.

Bernier asks further questions, in particular, with the example $\mathcal{A} = \mathbb{P} \setminus \{2\}$:

QUESTION 6.2.

(i) ◦ *Does $\zeta_{\mathcal{A}}$ have an analytic continuation?*

◦ *Can one perform Euler-Maclaurin summation on $\zeta_{\mathcal{A}}(\frac{1}{2} + it)$?, analytic, convergent?*

◦ *Where are the zeros?*

(ii) Is there a von Mangoldt type formula for the summatory Lambda function?

Well, let's head for some answers.

In the cofinite case, ie. if $\mathbb{P} \setminus \mathcal{A}$ is finite, $\zeta_{\mathcal{A}}(s) = \zeta(s) \prod_{p \in \mathbb{P} \setminus \mathcal{A}} (1 - p^{-s})$ (for $\mathrm{Re}\, s > 1$ and thus everywhere) and so $\zeta_{\mathcal{A}}$ is analytic everywhere but at its single pole $s = 1$. Its zeros are thus exactly those of ζ and additional ones on the imaginary axis: $1 - p^{-s} = 0$ iff $s \in \frac{2\pi}{\ln p} i\mathbb{Z} \subset i\mathbb{R}$. This reasoning does not work for non-cofinite sets $\mathcal{A}$,

In the finite case, ie. if $\mathcal{A} \cap \mathbb{P}$ is finite, $\zeta_{\mathcal{A}}$ is merely a finite product of factors $(1 - p^{-s})^{-1}$. These factors have no zeros and single poles at $\frac{2\pi}{\ln p} i\mathbb{Z}$.

In the cofinite case the summatory Lambda function should be

$$\vartheta^*_{\mathcal{A}}(x) = \sum_{\substack{p^k \leq x \\ p \in \mathcal{A} \cap \mathbb{P}}} \ln p = \vartheta^*(x) - \sum_{p \in \mathbb{P} \setminus \mathcal{A}} \ln p \left\lceil \frac{\ln x}{\ln p} \right\rceil$$

whose second part is more or less $\#(\mathbb{P} \setminus \mathcal{A}) \ln x$. The original von Mangoldt formula (3.2) turns into

$$\vartheta^*_{\mathcal{A}}(x) = x - \sum_{p \in \mathbb{P} \setminus \mathcal{A}} \ln p \left\lceil \frac{\ln x}{\ln p} \right\rceil - \sum_{\varrho} \frac{x^{\varrho}}{\varrho} - \underbrace{\frac{1}{2} \ln \left(1 - \frac{1}{x^2}\right)}_{= \sum_{n \geq 1} \frac{x^{-2n}}{-2n}} - \underbrace{\ln(2\pi)}_{= \frac{\zeta'(0)}{\zeta(0)}}.$$

A consideration of a direkt proof of (3.2) in this context should yield more general formulas.

Part IV
Algorithms

7. Prime counting algorithms

Relevant literature:

- Sorenson (1998).
- Pritchard (1982, 1994).

7.1. The sieve of Eratosthenes.

7.1.1. Basic sieve. Let's describe the version to sieve an interval $[x, x-L[:= \mathbb{Z}_{<x} \setminus \mathbb{Z}_{<x-L}$. We start with an array of bits indexed by the numbers in the interval. At first we set all bits to 1. Now, for each prime p less or equal to $\sqrt{x}$ for $k = 2p$ while $k \leq x$ in steps of p set the bit k to 0.

ALGORITHM 7.1. Sieve of Eratosthenes for an interval.

Input: Interval bound x, interval length L.
Output: Array $a[0..L]$ of bits such that $a[i] = 1$ iff $x - L + i$ is prime.

1. Initialize an array a of bits indexed from $0..L$ to all 1.
2. For p prime, $p \leq \sqrt{x}$ do 3–4
3. For $k = ((L - x) \operatorname{rem} p)$, $k < x$, k+= p do
4. $a[k] := 0$.
5. Return a

The classical sieve of Eratosthenes would have $L = x - 2$ and the primes for the loop are read of the array a. This algorithm needs

$$\text{time} \in \mathcal{O}\left(L \ln\ln x + \sqrt{x}\right), \quad \text{space} \in \mathcal{O}\left(L\right).$$

To speed up the algorithm we can decrease the memory requirements by a factor two by only storing odd numbers in the array. This amounts to predicting the effect of the smallest prime. Extending this we only consider numbers in $a + \mathbb{Z}_s$ with $s = \prod_{p \leq y}^{\mathbb{P}} p$, $a \in \mathbb{Z}_s^{\times}$. This saves a factor $\prod_{p \leq y}^{\mathbb{P}} (1 - \frac{1}{p})$. In practice, this is used with $y = 2$ to $y = 13$. ★¿★Is that less than Pritchard 1982, 1994?★?★

7.1.2. Segmented version. The first problem with the basic sieve is its memory usage. The segmented sieve resorts by cutting the target interval into smaller segments of length S, say $S|L$. The most costly remaining operation is the division for the inititial k in the inner loop. This would now have to be executed L/B times for every prime. To save this cost we introduce a table $k[p]$ carrying this information from block to block. Initialize $k[p] = (L - x) \operatorname{rem} p$, this will be the only divisions. For each p strike out $a[k[p] + tp]$ for $t = 0$ until $k[p] + tp$ exits the current block. This first overshooting value minus the block length B store in $k[p]$. Once a block has been fully processed output it and reuse the array for the next block. This algorithm needs

$$\text{time} \in \mathcal{O}\left(L \ln\ln x + \sqrt{x}\right), \quad \text{space} \in \mathcal{O}\left(S + \sqrt{x}\right).$$

7.1.3. Cache-optimized segmented version. To further increase the use of available resources we put the primes paired with the segment offset $k[p]$ in buckets attached to each segment. The bucket size is taylored to the size of the processor cache. For every prime in a bucket associated to the current block we strike out $a[k[p] + tp] := 0$ for t until $k[p] + tp$ leaves the current block. From the first overshooting value $k[p] + tp$ we subtract the segment size S until it is less than L and put the prime with this offset in a bucket associated to the corresponding segment. Actually, small primes might be treated differently as they touch every segment whereas larger primes jump over many segments. This algorithm still needs

$$\text{time} \in \mathcal{O}\left(L \ln\ln x + \sqrt{x}\right), \quad \text{space} \in \mathcal{O}\left(S + \sqrt{x}\right),$$

the gain lies in the optimized use of the processor's cache.

★¿★Pritchard's wheel sieve with time $\in \mathcal{O}\left(\frac{1}{\ln\ln x} L + ?\right)$, space $\in \mathcal{O}\left(?\right)$.★?★

7.2. The quadratic sieve.

Relevant literature:

- Atkin & Bernstein (2004).
- Galway (2004).

The quadratic sieve makes use of the fact that primes of the form $1 + 4k$ are a sum of two squares in an odd number of ways, whereas squarefree composite numbers are a sum of two squares in an even number of ways. More general we have the following theorem in Galway (2004) after Atkin & Bernstein (2004):

THEOREM 7.2. *Consider $n \in \mathbb{N}$ with $n \bmod 12 \in \mathbb{Z}_{12}^{\times} = 1, 5, 7, 11$. Choose the case with $C \ni n \bmod 12$:*

(i) $C = \{1, 5\}$, $\mathcal{R} = \{(u_1, u_2) \in \mathbb{R} \mid u_1 > u_2 > 0\}$, $Q(u_1, u_2) = u_1^2 + u_2^2$.

(ii) $C = \{7\}$, $\mathcal{R} = \{(u_1, u_2) \in \mathbb{R} \mid u_1, u_2 > 0\}$, $Q(u_1, u_2) = 3u_1^2 + u_2^2$.

(iii) $C = \{11\}$, $\mathcal{R} = \{(u_1, u_2) \in \mathbb{R} \mid u_1 > u_2 > 0\}$, $Q(u_1, u_2) = 3u_1^2 - u_2^2$.

Choose the case with $n \in C$ and let $\mathcal{P}(n) = \{(u_1, u_2) \in \mathbb{Z}^2 \cap \mathcal{R} \mid Q(u_1, u_2) = n\}$. Then

$$n \text{ prime} \iff n \text{ squarefree} \wedge \#\mathcal{P}(n) \text{ odd}. \qquad \square$$

This result can be turned into an algorithm to sieve an interval.

ALGORITHM 7.3.

Input: Interval bound x, interval length L.
Output: Array $a[0..L]$ of bits such that $a[i] = 1$ iff $x - L + i$ is prime.

1. Initialize an array a of bits indexed from $0..L$ to all 0.
2. For $(C, \mathcal{R}, Q)$ do 3–5
3. For $(u_1, u_2) \in \mathcal{R} \cap \mathbb{Z}^2 \cap Q^{-1}([x - L, x])$ do 4–5
4. $n := Q(u_1, u_2)$, $i := i - (x - L)$.
5. If $n \bmod 12 \in C$ then $a_i := 1 - a_i$
6. For $q = 3..\lfloor\sqrt{x}\rfloor$ do 7–8
7. For $m = \left\lceil \frac{x-L}{q^2} \right\rceil .. \frac{x}{q^2}$ do
8. $a_{mq^2} := 0$.

9. Return a

Some care is necessary to implement the u loop to avoid unnecessary multiplications (or even square roots). This algorithm needs

$$\text{time} \in \mathcal{O}\left(L \ln\ln x + \sqrt{x}\right), \quad \text{space} \in \mathcal{O}\left(L\right).$$

Segmentation can be employed here. The cache optimization is not clear to me… Galway (2004) proposes a dissection method. He dissects the u area and …

★¿★HERE★?★

7.3. The combinatorial method. Relevant literature:

- Oliveira e Silva (2006).
- Lagarias, Miller & Odlyzko (1985).
- Deleglise & Rivat (1996).

We now refrain from listing primes. The combinatorial method is the first that determines the number $\pi(x)$ of primes up to x without listing all those primes.

For any (positive) real x we consider the count

$$\pi_S^K(x) = \#\left\{ n \in \mathbb{N}_{\leq x} \;\middle|\; \begin{array}{r} \exists k \in K \colon \exists q_1, \dots, q_k \in S \cap \mathbb{P}\colon \\ n = q_1 \cdots q_k \end{array} \right\}$$

of natural numbers that for some $k \in K$ have exactly k prime factors from the set S. If S contains all (relevant) primes, we may omit it. Further, we abbreviate $\pi_S^k(x) = \pi_S^{\{k\}}(x)$ and $\pi_S^{\leq k}(x) = \pi_S^{\mathbb{N}_{\leq k}}(x)$. In particular, $\pi(x) = \pi_{\mathbb{R}}^1(x)$. Finally, we denote the a-th (natural) prime number by p_a, for instance, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$. As a special case, we set $p_0 := 1$.

Elsewhere used notation translates as follows: $\varphi(x, a) = \pi_{>p_a}^{\mathbb{N}}(x)$ counts the p_{a+1}-rough numbers, $\varphi_k(x, a) = \pi_{>p_a}^k(x)$ counts the p_{a+1}-rough numbers with exactly k prime factors.

Clearly, $\pi_{>A}^k(x) = 0$ as soon as $\operatorname{nextprime}(A+1)^k > x$, which is implied by $A^k \geq x$, and consequently

$$\pi_{>A}^{\mathbb{N}}(x) = \sum_{0 \leq k \leq \left\lfloor \frac{\ln x}{\ln A} \right\rfloor} \pi_{>A}^k(x). \tag{7.4}$$

Observe that $\pi_{>A}^0(x) = 1$ for $x \geq 1$ and $\pi_{>A}^1(x) = \pi(x) - \pi(A)$. Combining all that yields

$$\pi(x) = \pi_{>A}^{\mathbb{N}}(x) + \pi(A) - 1 - \sum_{2 \leq k \leq \left\lfloor \frac{\ln x}{\ln A} \right\rfloor} \pi_{>A}^k(x). \tag{7.5}$$

If we choose $A = \sqrt{x}$ then the sum is actually empty; though $\ln x / \ln A = 2$ allows the summand $\pi_{>A}^2(x)$, this term is actually zero as it counts numbers that are a product of two primes larger than A. So in that case we 'only' have to count $\pi_{>\sqrt{x}}^{\mathbb{N}}(x)$ and $\pi(\sqrt{x})$. The latter can be done using the standard sieve of Eratosthenes in time $\mathcal{O}^\sim\left(x^{\frac{1}{2}}\right)$. What about the former then? If we choose A smaller, say somewhere between $\sqrt[3]{x}$ and $\sqrt{x}$ still the sum has only the summand $\pi_{>A}^2(x)$. Lehmer used $A = \sqrt[4]{x}$ so he has to count $\pi_{>A}^3(x)$ as well.

7.3.1. Count $\pi_{>A}^2(x)$. We assume that $\sqrt[3]{x} \leq A \leq \sqrt{x}$. Now we can express

$$\begin{aligned} \pi_{>A}^2(x) &= \sum_{A<p\leq x}^{\mathbb{P}} \sum_{p \leq q \leq x}^{\mathbb{P}} \chi(pq \leq x) \\ &= \sum_{A<p\leq\sqrt{x}}^{\mathbb{P}} \sum_{p\leq q\leq x/p}^{\mathbb{P}} 1. \end{aligned}$$

Substituting the inner sum with $\pi(x/p) - (\pi(p) - 1)$ and noting that the second part merely is $\sum_{\pi(A)<r\leq\pi(\sqrt{x})}(r-1)$, we finally obtain the expression

$$\pi_{>A}^2(x) = \binom{\pi(A)}{2} - \binom{\pi(\sqrt{x})}{2} + \sum_{A<p\leq\sqrt{x}}^{\mathbb{P}} \pi(x/p). \tag{7.6}$$

To evaluate this, we could using the sieve of Eratosthenes obtain

- $\pi(A)$ with $\mathcal{O}^\sim(A)$ operations,
- $\pi(\sqrt{x})$ with $\mathcal{O}^\sim(\sqrt{x})$ operations, and
- each $\pi(x/p)$ by sieving up to x/A, cumulating it accounts for a total of again $\mathcal{O}(x/A)$ operations.

This is already more than anything before but we are limited to at least $\mathcal{O}(x/A)$ operations. Well, so we reuse (7.5) for $\frac{x}{p}$ noting that the sum collapses since $\sqrt[3]{x} \leq A < p \leq \sqrt{x}$, and so $A^2 \geq x^{\frac{2}{3}} > x/p$:

$$\pi\left(\frac{x}{p}\right) = \pi_{>A}^{\mathbb{N}}\left(\frac{x}{p}\right) + \pi(A) - 1.$$

So provided $A \geq \sqrt[3]{x}$ an efficient computation of $\pi_{>A}^{\mathbb{N}}(x)$ and of $\pi_{>A}^{\mathbb{N}}\left({}^{x}/{}_{p}\right)$ will give us an efficient computation for $\pi(x)$ in the end as far as we can see by now.

7.3.2. Count $\pi_{>A}^{\mathbb{N}}(x)$. We start with two observations. First, $\pi_{>A}^{\mathbb{N}}(x)$ counts exactly those numbers that are coprime to $P_A := \prod_{p\leq A}^{\mathbb{P}} p$. This condition is periodic, per interval we count $\pi_{>A}^{\mathbb{N}}(P_A) = \varphi(P_A) = \prod_{p\leq A}^{\mathbb{P}}(p-1)$ numbers, and in case P_A is small with respect to x we can thus benefit from

$$\pi_{>A}^{\mathbb{N}}(x) = \left\lfloor \frac{x}{P_A} \right\rfloor \varphi(P_A) + \pi_{>A}^{\mathbb{N}}\left(x - \left\lfloor \frac{x}{P_A} \right\rfloor P_A\right). \tag{7.7}$$

To benefit from this we need to have the values $\pi_{>A}^{\mathbb{N}}(x)$ precomputed for a suitably small values for x.

Second, every number n below x has a smallest prime factor $p_{\min}(n)$. For $\pi_{>p_a}^{\mathbb{N}}(x)$ the number n is counted if $p_{\min}(n) > p_a$. If instead we count all numbers with $p_{\min}(n) \geq p_a$, or equivalently $p_{\min}(n) > p_{a-1}$, we have to subtract (that's sieving) those with $p_{\min}(n) = p_a$. But these latter are of the form $n = p_a m$ with $m \leq \frac{x}{p_a}$ and $p_{\min}(m) > p_{a-1}$. This explains

$$\pi_{>p_a}^{\mathbb{N}}(x) = \pi_{>p_{a-1}}^{\mathbb{N}}(x) - \pi_{>p_{a-1}}^{\mathbb{N}}\left(\frac{x}{p_a}\right). \tag{7.8}$$

Directly or by (7.7), $\pi_{>1}^{\mathbb{N}}(x) = \lfloor x \rfloor$. Together with (7.8) this leads to the inclusion-exclusion formula

$$\pi_{>p_a}^{\mathbb{N}}(x) = \sum_{\substack{n\leq x \\ p_{\min}(n)\leq p_a}} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor \tag{7.9}$$

Obviously, $\pi^{\mathbb{N}}_{>p_a}(x) = 1$ if $p_a \le x < p_{a+1}$ which allows us to stop the recursion a step earlier and actually to avoid computations with result 0. Altogether one can turn this into an algorithm using

$$\text{time} \in \mathcal{O}\left(\frac{x}{\ln x}\right), \text{space} \in \mathcal{O}\left(\frac{\sqrt{x}}{\ln x}\right).$$

To reduce this Lagarias, Miller & Odlyzko (1985) try to keep the computation tree smaller. They set $A = \alpha\sqrt[3]{x}$ with a carefully chosen value $\alpha \ge 1$, fix parameters c and $z = x/A = \alpha^{-1}x^{\frac{2}{3}}$, and on computation of $\pi^{\mathbb{N}}_{>p_b}(x')$ stop the use of the recursion (7.8) if we reach either

1. an ordinary leave: $b = c$ and $x' \ge z$, or
2. a special leave: $x' < z$.

As mentioned earlier, we can use (7.7) when the period $P_{p_b} = \prod^{\mathbb{P}}_{p \le p_b} p$ is small enough. This will determine the choice of c. On the other hand, we can hope counting $\pi^{\mathbb{N}}_{>p_b}(x')$ directly if x gets small, the chosen threshold is z.

Before we get to the details let us describe how to index the computation tree. The node for $\pi^{\mathbb{N}}_{>p_b}(x')$ in the tree either is defined as a leave or it has two childs, one for $\pi^{\mathbb{N}}_{>p_{b-1}}(x')$ and the other for $-\pi^{\mathbb{N}}_{>p_{b-1}}\left(\frac{x'}{p_b}\right)$. Each node in the tree thus is of the form $\mu(n)\pi^{\mathbb{N}}_{>p_b}\left(\frac{x}{n}\right)$ with n being a product a subset of $\{p_{b+1}, \dots, p_a\}$. In an ordinary leave we thus have $x' = {}^{x}/_{n}$ with $n \le x/z = A$. In a special leave we have $n > A$. See Figure 7.1 for an example.

7.3.3. Ordinary leaves. The aim here is to deal with all ordinary leaves simultaneously. Their indices are precisely given by those natural numbers with $n \le A$, $\mu(n) \ne 0$, $p_{\min}(n) > p_c$. So altogether the special leaves contribute to $\pi^{\mathbb{N}}_{>p_a}(x)$ the value

$$S_0 = \sum_{\substack{n \le A \\ p_{\min}(n) > p_c}} \mu(n)\pi^{\mathbb{N}}_{>p_c}\left(\frac{x}{n}\right). \tag{7.10}$$

A modification of the sieve of Eratosthenes gives us for $n \in \mathbb{N}_{<A}$ the value $[\mu(n), p_{\min}(n)]$. For each $n \in \mathbb{N}_{<A}$ with $p_{\min}(n) > p_c$ we now compute $\pi^{\mathbb{N}}_{>p_c}\left(\frac{x}{n}\right)$ using (7.7). Notice that there are at most A ordinary leaves. So this amounts to a total time of $\mathcal{O}(A \ln\ln A)$, most of which is needed for the sieving. Actually, c is considered constant here, say $c = 7$, since the precomputation of $\pi^{\mathbb{N}}_{>p_c}(x')$ for $x' \in \mathbb{N}_{<P_{p_c}}$ otherwise is too costly. Well, maybe we can allow a larger c if we employ (7.8) for $\pi^{\mathbb{N}}_{>b}(x')$ with $b > 7$, say.

7.3.4. Special leaves. Subdivide the special leaves $\pi^{\mathbb{N}}_{>b}\left(\frac{x}{n}\right)$ once more:

- First case, $p_{b+1}^2 \le A$: Cheap.
- Second case, $p_{b+1}^2 > A$. Here $n = p_{b+1}p_d$ for some $\pi(\sqrt{y}) < d \le a$. Subdivide into
  - Trivial leaves: Each trivial leave contributes 1. So we merely need to count them. That amounts to counting primes. Some can be clustered, some are sparse.
  - Easy leaves: Can be computed by counting primes using (7.4) with summands $k = 0$ and $k = 1$ only (or equivalently (7.5) with empty sum).
  - Hard leaves: For these you have to compute $\pi^{\mathbb{N}}_{>b}\left(\frac{x}{n}\right)$ the hard way.

7.4. The analytical method. Relevant literature:

- Lagarias & Odlyzko (1987).
- Galway (2004).

Riemann's identity

$$\pi_0^*(x) = \sum_{p^k \le x} \frac{1}{k} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s} \ln\zeta(s)\, \mathrm{d}s \tag{7.11}$$

with $a > 1$ arbitrary may also be used to compute $\pi(x)$ via (1.7). The integral here has to be understood as a path integral along the line with $\Re s = a$. However, it turns out that a good approximation of the integral on the right is difficult to obtain since on the line $\Re s = a$, both x^s and $\ln\zeta(s)$ oscillate, and the integral is not absolutely convergent. Lagarias & Odlyzko (1987) say that the convergence can be estimated but not in a sufficient way. Instead they resort to changing the formulae:

$$\sum_{p^k \le x} \frac{c(p^k)}{k} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\ln\zeta(s)\, \mathrm{d}s \tag{7.12}$$

where $c(u)$ and $F(s)$ are a Mellin transform pair, that is,

$$c(u) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)u^{-s}\, \mathrm{d}s\,, \tag{7.13}$$

$$F(s) = \int_0^\infty c(u)u^{s-1}\, \mathrm{d}u\,. \tag{7.14}$$

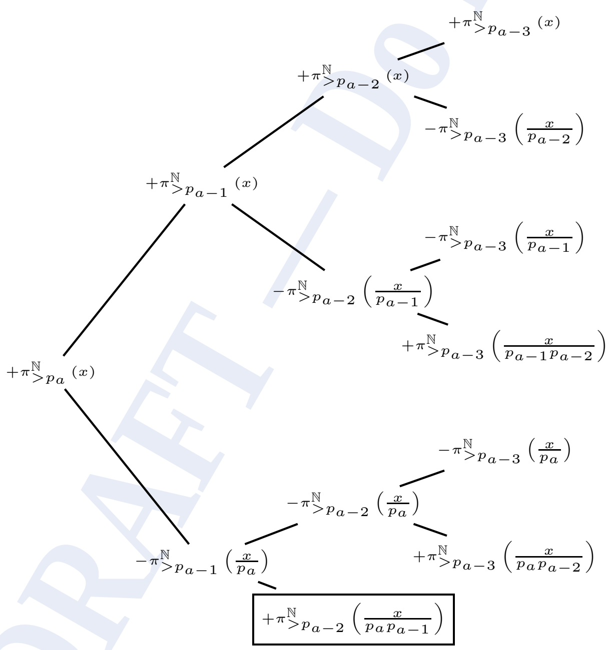

Figure 7.1: Computation tree when $c = a - 3$ and $z = \frac{x}{p_a p_{a-2}}$. Most leaves are ordinary, the boxed one is special.

If we use $c = c_0$ with $c_0(u) = 1$ for $u < x$ and $c_0(u) = 0$ for $u > x$ we obtain (7.11). However, then $F(s) = \frac{x^s}{s}$ which makes the intgral difficult to compute. Instead, we modify c_0 in an interval $]x - y, x[$ in a way that makes F decrease rapidly as $|\Im s| \to \infty$. This then allows a numerical evaluation of the integral. Of course, we do not obtain $\pi(x)$ or $\pi^*(x)$ then but the difference to $\pi^*(x)$ only depends on the prime powers within the interval $]x - y, x[$ on which c differs from c_0:

$$\pi^*(x) - \sum_{p^k \le x} \frac{c(p^k)}{k} = \sum_{x-y<p^k\le x} \frac{1-c(p^k)}{k}. \tag{7.15}$$

This can be computed via sieving, say, in about y steps, which is ok provided that y is not too large. Lagarias & Odlyzko (1987) use a c for which the contribution of the region with $|\Im s| \ge T$ is negligible for $T \ge \frac{x}{y}$. Computing the integral numerically now requires $\mathcal{O}\left(Tx^{\varepsilon}\right)$ evaluations of the integrand. This boils down to evaluate ζ at these points provided F is sufficiently easy to compute. Odlyzko & Schönhage (1988) describes how multi-evaluation of ζ at values $a + it$ for $u \le tu + u^{\beta}$ can be done within $\mathcal{O}\left(u^{\frac{1}{2}}\right)$ time and $\mathcal{O}\left(u^{\beta}\right)$ space for any $\beta \in \left[0, \frac{1}{2}\right]$. This leads to an algorithm using

$$\text{time} \in \mathcal{O}\left(x^{\frac{3-2b}{5}+\varepsilon}\right), \quad \text{space} \in \mathcal{O}\left(x^{b+\varepsilon}\right)$$

where we can choose $b \in \left[0, \frac{1}{4}\right]$.

Galway (2004) proceeds similarly but uses a different kernel function. Let

$$\varphi(u; x, \lambda) = \frac{1}{2} \operatorname{erfc}\left(\frac{\ln \frac{u}{x}}{\sqrt{2}\lambda}\right) \tag{7.16}$$

based on the complementary error function $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \exp(-r^2)\, \mathrm{d}r$. The Mellin transform of this kernel function is

$$\Phi(s; x, \lambda) = \exp\left(\frac{\lambda^2 s^2}{2}\right) \frac{x^s}{s}. \tag{7.17}$$

Define $\pi^*(x;\lambda) = \sum_{p^k \le x} \frac{1}{k} \varphi(p^k; x, \lambda)$. For $\lambda = 0$ this is precisely $\pi^*(x)$ with its standard definition. The inverse Mellin transformation provides us with the integral description

$$\pi^*(x;\lambda) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Phi(s; x, \lambda) \ln \zeta(s)\, \mathrm{d}s\,, \tag{7.18}$$

that turns out to be well to approximate. On the other hand the difference to $\pi^*(x)$ can be computed directly,

$$\pi^*(x) - \pi^*(x;\lambda) = \sum_{p^k} \frac{1}{k} \left(\varphi(p^k; x, 0) - \varphi(p^k; x, \lambda)\right). \tag{7.19}$$

Other than with the kernel using by Lagarias & Odlyzko (1987) this sum is not finite, however its major contribution come from finitely many primepowers close to x. (Galway 2004, §2.5) argues that this kernel is optimal.

To make everything work one needs to bound errors for the approximation of the integral (7.18) and the approximation in the sum (7.19). Choosing parameters such that the common error is comfortably smaller than $\frac{1}{2}$, the computation can be run. Finally, the computation of the sum can be combined with the final step: $\pi(x) = \pi^*(x) - \sum_{k\ge 2} \frac{1}{k} \pi\left(x^{\frac{1}{k}}\right)$.

7.5. Counting modulo 2. Relevant literature:

- Lifchitz (2001).

8. Algorithm to evaluate Riemann's ζ function

Relevant literature:

- Odlyzko & Schönhage (1988).

9. Algorithm to approximate the number $\Psi(x, y)$ of y-smooth integers below x

See Hunter & Sorenson (1997). Based on Hildebrand & Tenenbaum (1986) they essentially only need to compute values and derivatives of $\zeta_{\le y}(s) := \prod_{p \le y} (1 - p^{-s})^{-1}$ to obtain $\Psi(x, y)$ up to a relative error of order $\mathcal{O}\left(\frac{\ln y}{\ln x} + \frac{y}{\ln y}\right)$. This has been further improved in Parsell & Sorenson (2006).

Part V

Diverse

10. A monotonicity property

Sondow & Dumitrescu (2010) claim that in any zero-free, right open half-plane ξ is isotone on any horizontal line. They do not claim that given $t \in \mathbb{R}$ such that there is no zero of ξ on the half line $L = \mathbb{R}_{>\frac{1}{2}} + it$ the function $\mathbb{R}_{>\frac{1}{2}} \to \mathbb{R},\ s \mapsto |\xi(s+it)|^2$ is isotone.

If that is true then also the Riemann hypothesis holds: Since ξ is analytic the function $\alpha\colon \mathbb{R}_{>\frac{1}{2}} \times \mathbb{R} \to \mathbb{R},\ (s,t) \mapsto \frac{\partial}{\partial s}|\xi(s+it)|^2$ is continuous. Assume that RH is false. Then we have $\varrho = \sigma + it$ with $\xi(\varrho) = 0$ and $\sigma > \frac{1}{2}$. Since $|\xi|^2 \geq 0$ we have $\alpha(\sigma, t) = 0$ and there is a small $\varepsilon \in \left]0, \sigma - \frac{1}{2}\right[$ such that $\alpha(\sigma - \varepsilon, t) < 0$. Now however there exists a $\delta > 0$ such that $\alpha(\sigma - \varepsilon, t+\delta) < 0$ but there is no ξ zero on the line $\mathbb{R}_{>\frac{1}{2}} + i(t+\delta)$ since the zeros of ξ are discrete. But this contradicts the above statement.

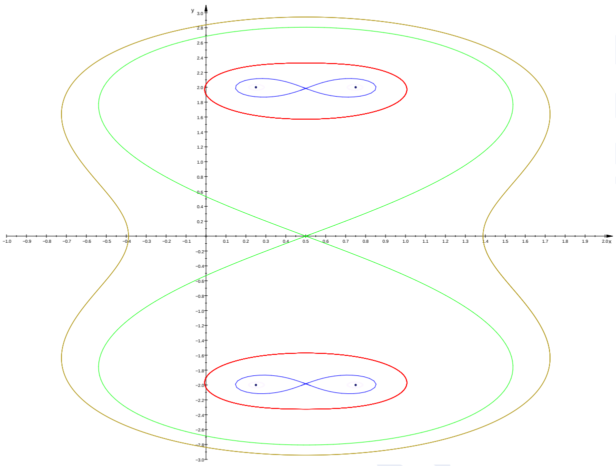

Figure 10.1: Contour lines of $\prod |s - \varrho_i|$ for the depicted four points ϱ_i

The idea of the above comes from the product formula for ξ:

(10.1)
$$\begin{aligned}\xi(s) &= \frac{1}{2}\prod_{\varrho}\left(1 - \frac{s}{\varrho}\right)\\ &= c\prod_{\varrho}\left(1 - \frac{s-\frac{1}{2}}{\varrho - \frac{1}{2}}\right)\end{aligned}$$

Actually, ξ is isotone on $\left[\frac{1}{2}, 1\right]$ (Proof?). Now, any zero ϱ with $\mathrm{Re}(\varrho) > \frac{1}{2}$ causes the absolute value $\left|1 - \frac{s}{\varrho}\right| = \frac{|s-\varrho|}{|\varrho|}$ to decrease and no finite number of roots on the critical line can repair this on a horizontal line close to ϱ. Of course, this factor has to be combined with its mirrored versions:

$$\left|\left(1 - \frac{s}{\varrho}\right)\left(1 - \frac{s}{\bar\varrho}\right)\left(1 - \frac{s}{1-\varrho}\right)\left(1 - \frac{s}{1-\bar\varrho}\right)\right|$$

If ϱ is large this is close to

$$\begin{aligned}&\left|1 - s\left(\frac{1}{\varrho(1-\varrho)} + \frac{1}{\bar\varrho(1-\bar\varrho)}\right)\right|\\ &= \left|1 - s\left(\frac{1}{(\sigma+it)(1-\sigma-it)} + \frac{1}{(\sigma-it)(1-\sigma+it)}\right)\right|\end{aligned}$$

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A. TODO

- ★5.1:6 “Note that the characters of modulus k form a group and linearly span all k-periodic functions $\mathbb{N} \to \mathbb{C}$.” is wrong! Find the correct statement!
- ★5.2:6 Weil’s explicit formula
- ★7.1:8 Is that less than Pritchard 1982, 1994?
- ★7.2:8 Pritchard’s wheel sieve with time $\in \mathcal{O}\left(\frac{1}{\ln\ln x} L+?\right)$, space $\in \mathcal{O}\left(?\right)$.
- ★7.3:9 HERE.

Notes

[i]More generally: If $c(x,1) = x$, $c(c(x,n),m) = c(x,nm)$, $g(c(x,n)) = 0$ for $n > x$, and $f(x) = \sum_{n \leq x} g(c(x,n))$ then

$$g(x) = \sum_{n \leq x} \mu(n) f(c(x,n)).$$

For example $c(x,n) = x^{1/n}$ with any g that vanishes for arguments less than 2 fulfills the requirements. (Since $\exp(\frac{\ln x}{x}) < 2$ is equivalent to $x < 2^x$ which is true for $x > 1$, the condition for g is also fulfilled.)

PROOF. Start with the right hand side of the claim:

$$\begin{aligned}
\sum_{n\leq x} \mu(n) f(c(x,n)) &= \sum_{n\leq x} \mu(n) \sum_{m\leq x} g(c(c(x,n),m)) \\
&= \sum_{n\leq x} \sum_{m\leq x} \mu(n) g(c(x,nm)) \\
&= \sum_{k\leq x} \underbrace{\sum_{\substack{n,m\leq x\\ nm=k}} \mu(n)}_{=\begin{cases}1 & \text{if } k=1,\\ 0 & \text{otherwise.}\end{cases}} g(c(x,k)) \\
&= g(c(x,1)) = g(x).
\end{aligned}$$

We use that $g(c(x,k)) = 0$ where we restrict the sum over k to $k \leq x$. ☐

[ii]Most sources define

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{\exp(t)}{t}\,\mathrm{d}t \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$$

Some (including MuPAD 3) define

$$\begin{aligned}
E_1(z) &= \int_{z}^{\infty} \frac{\exp(-t)}{t}\,\mathrm{d}t \qquad \text{if } z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \\
&= \int_{1}^{\infty} \frac{\exp(-uz)}{u}\,\mathrm{d}u \qquad \text{for } \Re x \geq 0
\end{aligned}$$

Unfortunately, these definitions differ in particular at those values where we need it. And since $\mathrm{Ei}(x) = -\mathrm{E}_1(-x)$ for $x > 0$ also the branch cut is on different sides when translated. This leads to $\mathrm{Ei}(x) = -E_1(-x) \mp i\pi$ for $x < 0$.

We encounter the exponential integral in form of the logarithmic integral $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{t}$. We can express $\mathrm{Li}(x) = \mathrm{Ei}(\ln x) - \mathrm{Ei}(\ln 2) = -\mathrm{E}_1(-\ln x) + \mathrm{E}_1(-\ln 2)$, which is only save for $x \in \mathbb{R}_{\neq 0}$.

[iii]A guess for ϑ:

$$\begin{aligned}
\vartheta_0(x) = \sum_{n\geq 1} \frac{\mu(n)}{n} x^{\frac{1}{n}} - \sum_{\varrho} \sum_{n\geq 1} \frac{\mu(n)}{n} \frac{x^{\varrho/n}}{\varrho} \\
- \sum_{n\geq 1} \frac{\mu(n)}{n} \frac{1}{2} \ln\left(1 - \frac{1}{x^{2/n}}\right) - \sum_{n\geq 1} \frac{\mu(n)}{n} \ln(2\pi).
\end{aligned}$$

[iv]Examples:

- The principal character χ_0.
- The quadratic character $\left(\frac{\cdot}{k}\right)$ modulo k.
- Suppose $\alpha \in \mathbb{Z}_k^\times$ generates $\mathbb{Z}_k^\times$, and $\zeta \in \mathbb{C}^\times$ is a $\varphi(k)$-th primitive root of unity. Then any period-k character χ is given by $\chi(\alpha^i) = \zeta^{si}$ for some $s \in \mathbb{Z}_{\varphi(k)}$. In particular, the quadratic character is given by $\left(\frac{\alpha^i}{k}\right) = (-1)^i = \zeta^{\frac{\varphi(k)}{2}i}$.

Contents