Bonn-Aachen International Center for Information Technology
Cryptography and Game Theory — Raekow, Ziegler
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1 Equilibrium Computation

Of course we want to compute the NE of a given game G. Therefor we need first of all a new condition for a NE. We will also discuss this condition in an example.

1.1 Minmax for zerosum games

In this part of the lecture we consider zerosum games for two players, i.e. games where the sum of the utility function of all players is zero (for all possible strategies). For example the matching pennies game

	H	T
H	1	-1
T	-1	1

If the first player win 1$ the secend player loose one and vice versa.

For such a two player game we define two values

$$\overline{v_i} = \min_{s_{1-i}} \max_{s_i} u_i(s_i, s_{1-i}) \begin{bmatrix} \text{value} \\ \text{when} \\ \text{moving} \\ \text{second} \end{bmatrix}$$

and

$$\underline{v_i} = \max_{s_i} \min_{s_{1-i}} u_i(s_i, s_{1-i}) \begin{bmatrix} \text{value} \\ \text{when} \\ \text{moving} \\ \text{first} \end{bmatrix}.$$

The value $\overline{v_i}$ is the maximal gain if the player i can react to the strategy of player $1-i$. In contrast to the value $\underline{v_i}$, which is the maximal gain if player i is moving second, then player $1-i$ can react to player i's strategy. In the MP game is $\overline{v_i} = 1$ and $\underline{v_i} = -1$ if we consider only pure strategies.

Proposition 1.

$$\underline{v_i} \leq \overline{v_i} = \underline{v_{1-i}}$$

Proof. For all s_i and s^*_{1-i} hold

$$\min_{s_{1-i}} u_i(s_i, s_{1-i}) \leq u_i(s_i, s^*_{1-i}).$$

So it holds also if we consider the maximum over all s_i on both sides

$$\max_{s_i} \min_{s_{1-i}} u_i(s_i, s_{1-i}) \leq \max_{s_i} u_i(s_i, s^*_{1-i}).$$

Finally we take the minimum over all i on both sides

$$\min_{s^*_{1-i}} \underbrace{\max_{s_i} \min_{s_{1-i}} u_i(s_i, s_{1-i})}_{\underline{v_i}} \leq \underbrace{\min_{s^*_{1-i}} \max_{s_i} u_i(s_i, s^*_{1-i})}_{\overline{v_i}}.$$

For the first part we are done since there is no term s^*_{1-i} on the left hand side (we can ignore the minimum over s^*_{1-i}).

The second part is a simple calculation

$$\begin{aligned}
\overline{v_i} &= \min_{s_{1-i}} \max_{s_i} u_i(s_i, s_{1-i}) \\
&= \min_{s_{1-i}} \max_{s_i} -u_{1-i}(s_i, s_{1-i}) \\
&= \min_{s_{1-i}} \bigl(-\min_{s_i} u_{1-i}(s_i, s_{1-i})\bigr) \\
&= -\max_{s_{1-i}} \min_{s_i} u_{1-i}(s_i, s_{1-i}) = -\underline{v_i}
\end{aligned}$$

□

Computing $\underline{v_i}$ and $\overline{v_i}$ is an optimization problem that can be solved by linear programming. Later we will see efficient algorithms that use this fact. We have the following theorem (without a proof in this lecture).

Theorem 2 (Minmax, von Neumann 1928)**.**

$$\underline{v_i} = \overline{v_i}$$

1.2 Bimatrix games

Bimatrix games are aka finite two player strategic games. Player 1 can choose a row in $M = \{1, \ldots, m\}$ and player 2 a column in $N = \{1, \ldots, n\}$. The payoff of the players is given by two matrices $A, B \in \mathbb{R}^{M \times N}$, w.l.o.g. with non-negative entries. The (mixed) strategies are column vectors $x \in \mathbb{R}^M$ and $y \in \mathbb{R}^N$ where $|x| = |y| = 1$ and $x, y \geq 0$. With this notation player 1's expected payoff is $u_1(x, y) = x^T Ay$.

In the following part we will always consider the example where the two following matrices are given

$$A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}.$$

A best response to the mixed strategy y of player 2 is a mixed strategy x that maximizes player 1's payoff $u_1(x, y) = x^T Ay$. E.g. let player 2 play the pure strategy $y = \binom{1}{0}$. If we take A of our example we get $Ay = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$. If we have the notation $x^T = (x_1, x_2, x_3)$ player 1's payoff is

$$u_1(x, y) = x^T Ay = 3x_1 + 2x_2.$$

That means a best response of player 1 to y is the strategy $x = (1, 0, 0)$.

Similar a best response of player 2 is defined.

Of course we have the fact that a NE is a pair of strategies that are best responses to each other and vice versa. In the next proposition we formulate a condition for best responses.

Proposition 3 (best response condition)**.** *x is a best response to y if and only if for all $k \in \operatorname{supp}(x)$, we have*

$$(Ay)_k = u = \max_{i \in M} (Ay)_i. \tag{$*$}$$

Proof. $(Ay)_i$ is the expected payoff to player 1 when playing the (pure) strategy i against y. Then

$$\begin{aligned} x^T A y &= \sum_{i \in M} x_i (Ay)_i \\ &= \sum_{i \in M} x_i \Big(u - \big(u - (Ay)_i\big) \Big) \\ &= \underbrace{\sum_{i \in M} x_i}_{=1} u - \sum_{i \in M} x_i \big(u - (Ay)_i\big) \\ &= u - \sum_{i \in M} \underbrace{x_i}_{\geq 0} \underbrace{\big(u - (Ay)_i\big)}_{\geq 0} \end{aligned}$$

We have $x^T A y \leq u$ and $x^T A y \leq 0$ iff

$$x_k > 0 \quad \Rightarrow \quad u = (Ay)_k.$$

$\square$

In our example we have a NE in pure strategies (support size $(1, 1)$):

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now we try to find NE in mixed strategies. At first we consider the support size $(1, 2)$ and $(2, 1)$: any pure strategy has a unique pure strategy as best response, so there are no NE with the desired support size and we need to mix at least two strategies each.

So let's investigate the support size $(2, 2)$: player 2 has only one possibility to choose 2 columns, but player 1 has three possibilities to chose two rows. So we have three cases:

Player 1 mixes row 1 and 2 To make player 2 indifferent between the two columns we compute

$$3x_1 + 2x_2 = 2x_1 + 6x_2 \quad \text{and} \quad x_1 + x_2 = 1.$$

The second equality holds as x is a mixed strategy. The result is $x_1 = 4/5$ and $x_2 = 1/5$. So the expected payoff to player 2 is $x^T B = (2.8, 2.8)$. To make player 1 indefferent between the two rows we have the equalities

$$3y_1 + 3y_2 = 2y_1 + 5y_2 \quad \text{and} \quad y_1 + y_2 = 1.$$

The result is $y_1 = 2/3$ and $y_2 = 1/3$. So the expected payoff to player 1 is $Ay \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$ and $(*)$ holds. So have found a NE.

Player 1 mixes row 1 and 3 To make player 1 indifferent between the two rows, we have the equalities

$$3y_1 + 3y_2 = 6y_2 \quad \text{and} \quad y_1 + y_3 = 1.$$

We get the result $y_1 = 1/2$ and $y_3 = 1/2$. But the expected payoff is $\begin{pmatrix} 3 \\ 3.5 \\ 3 \end{pmatrix}$, $(*)$ does not hold. Also to make player 2 indifferent between two columns leads to a contradiction.

$$3x_1 + x_3 = 2x_1 + x_3 \quad \text{and} \quad x_1 + x_3 = 1$$

has solution $x_1 = 2$ and $x_3 = -1$. In contradiction to $0 \leq x_i \leq 1$.

Last case is left as an exercise.