Bonn-Aachen International Center for Information Technology
Cryptography and Game Theory — Raekow, Ziegler
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1 Game Theory Part

A =

$$\begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix}$$

B =

$$\begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}$$

Best respons criterion: x is a best response to y iff for all $k \in supp(x)\, x^T Ay = u = \max i \in M(Ay)_i$

Consider support (2, 3) and (4, 5) To make player 1 indifferent between 2 and 3: $2y_4 + 5y_5 = 0y_4 + y_5\ y_4 + y_5 = 1 \Rightarrow (y_4, y_5) = (\frac{1}{3}, \frac{2}{3})$. The expected payoff for player 1 is $(3, 4, 4)^T$, so (*) is satisfied.

To make player 2 indifferent between 4 and 5 we solve $2x_2 + 3x_3 = 6x_2 + x_3\ x_2 + x_3 = 1$ $\Rightarrow (x_2, x_3) = (\frac{1}{3}, \frac{2}{3})$ With expected payoff $(\frac{8}{3}, \frac{8}{3})$ so (*) is satisfied.

What about support size (1, 2, 3) and (4, 5)?

A mixed strategy of player 2 is already uniquely determined by equalizing expected payoffs for *two* pure strategies of player 1. Then the payoff for the third strategy is different.

Def.: A two-player game is *non degenerated* if no mixed strategy of support size k has more than k best pure responses.

Cor.: In any NE (x, y) of a non-degenerate bimatrix game, the support sizes of x and y are equal.

Algo: Equilibria by support enumeration

Input: a nondegenerate bimatrix game given by $A, B \in R^{M \times N}$

Output: list of all NE of the game

- $NE = []$
- For each $k \in 1, ..., minm, n$:
- For each pair (I, J) of k-sized subsets of M and N:
- Solve
  $\sum i \in Ix_i b_{ij} = v$ for $j \in J$, $\sum x_i = 1$
  $\sum j \in Ja_{ij} y_j = u$ for all $i \in I$, $\sum y_i = 1$

- $Y\vec{x} \geq \vec{0}$, $\vec{y} \geq \vec{0}$ and (*) holds for x and analogously y
- NE.append((x, y))
- return NE

1.1 Equilibria via labeled polytopes

Notation:

- *affine combination* of points z_i : $\sum \lambda_i z_i$, with scalars λ_i, s.t. $\sum \lambda_i = 1$.
- *convex combination* of points z_i : $\sum \lambda_i z_i$, with scalars λ_i, s.t. $\sum \lambda_i = 1$ and $\lambda_i \geq 0 \forall i$.
- a set of points is *convex*, if closed under convex combinations.
- d points are *affinely independent*, if none of them is an affine combination of the $d-1$ others
- a convex set has dimension d if it has d+1 affinely independent points, but no more.
- a *polyhedron* $P \leq R^d$ is defined by $P = z \in R^d | Cz \leq q$, where C is a matrix and q is a vector.
- a polyhedron is called *full-dimensional* if $dim(P) = d$
- a polyhedron is called a *polytope* if it is bounded.
- a *face* of P is defined by $F = z \in P | c^T z = q_0$ s.t. $c^T z \leq q_0$ for all z in P.
- a 0-dim face is called a *vertex*
- a 1-dim face is called a *edge*
- a 2-dim face is called a *facets*

Fact: Any non-empty face $F \leq P$ can be obtained by turning some inequalities difining P into equalities. These are called binding inequalities for F. $F = z \in P | c_i z = q_i fori \in I$

A facet is characterized by a single binding inequality (which is not redundant).

The best-response polyhedron consists of a player's mixed strategies and the expected payoffs (+larger values) of the other player.

Example: For player 2 in the previous game $\bar{Q} = \{(y_4, y_5, u)|$

$$\begin{pmatrix} 3y_4 + 3y_5 \leq u \\ 2y_4 + 5y_5 \leq u \\ 0y_4 + 6y_5 \leq u \\ y_4 \geq 0 \\ y_5 \geq 0 \\ y_4 + y_5 = 1 \end{pmatrix}$$

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