1 Game Theory

1.1 Recall

Best-response polyhedron $\bar{P} = (x, v) \in R^M x R_+ \geq vec 0, B^T x \leq \vec{1}_v, \vec{1} x = 1$ $\bar{Q} = (y, u) \in R^N x R_+ Ay \leq \vec{1}_u, y \geq vec 0$

Facets are either own strategies (e.g. 4, 5 in the case of $\bar{Q}$) or strategies of the other player (e.g. 1, 2, 3 in the case of $\bar{Q}$)

In particular, facets 1, 2 and 3 indicate best responses along with payoffs, e.g. for $y_4 \geq \frac{2}{3}$ a best response is 1. Facets 4 and 5 indicate when the respective own strategy has probability 0.

Def.: A point $(y, u) \in \bar{Q}$ has label $k \in M \cup N$ if the $k$th inequality in the definition of $\bar{Q}$ is binding.

For $k = i \in M$ this means $(Ay)_i = u$ (meaning strat $i$ is a best response to $y$) For $k = j \in N$ this means $y_j = 0$ (meaning $y \notin supp(y)$)

Exp.: $(\frac{2}{3}, \frac{1}{3}, 3) \in \bar{Q}$ has labels 1, 2.

Prop: A NE is a pair $(x, y)$ of mixed strategies, such that with corresponding payoffs $u = x^T Ay, v = x^T By$, the pair $((x, v), (y, u)) \in P \times Q$ is completely labeled, i.e. every label appears as a label of $(x, v)$ or $(y, u)$.

Simplify under the assumption that $A$, $B^T$ are nonnegative and have no zero columns (nonzero payoffs). $\bar{P} = x \in R^M | x \geq vec 0, B^T x \leq \vec{1} \bar{Q} = y \in R^N | Ay \leq \vec{1}, y \geq vec 0$

Remark: New Notions! $x$ and $y$ are now scaled (by $u$ and $v$) versions of those from above. So they don’t sum up to 1 anymore, but the ratio is still the same.

$P \rightarrow \bar{P} (x, v) \rightarrow \frac{1}{v} v = (1^T x)^{-1} \leftarrow x (x \ast v, v)$

This correspondence maps facets to facets and in particular labelled points to points with the same labels.

Cor.: A NE is a completely labeled pair $(x, y)$, where $x \in P \setminus \vec{0}, y \in Q \setminus \vec{0}$

Exp.: Algo:

Equilibrium by vertex enumeration

Input: nondegenerate bimatrix game

Output: all NE of that game

1. NE = []

2. for all vertices $x \in P \setminus \vec{0}$ and all vertices $y \in Q \setminus \vec{0}$: if $(x, y)$ is completely labeled: NE.append $(\frac{x}{1^T x}, \frac{y}{1^T y})$

3. return NE
For \( n = m \):

\[ \# \text{ support pairs } \approx 4^n \]

\[ \# \text{ vertex pairs } \approx 2.6^n \]

Algorithm: LEMKE - HOWSON

Input: nondegenerate bimatrix game

Output: one(!) NE

1. \((x, y) \leftarrow (vec0, vec0) \in PxQ\# \) artificial equilibrium completely labeled

2. Pick a label \( k \in M \cup N \) and drop it from \((x, y)\). Call new vertex \((x', y')\)

3. While \((x', y')\) is not completely labeled alternatingly drop duplicate labels and call resulting new vertex \((x', y')\).

4. return \(\left(\frac{x'}{vec1^T x'}, \frac{y'}{vec1^T y'}\right)\)

Prop.: Lemke-Howson terminates and returns a NE.

Proof: For a fixed random choice \( k \) the traversal graph of visited vertices has only vertices of degree 1 (equilibria) 2 (drop in P or in Q). The endpoints are completely labeled. Their number is even and subtracting the artificial one yields.

Cor.: A nondegenerate bimatrix game has an odd number of NE.