Bonn-Aachen International Center for Information Technology
Cryptography and Game Theory
IPEC Winter 2012
Raekow, Ziegler

Scribe(s): David Möller
Lecture 14, Date: 22.03.2012

1 Game Theory

1.1 Recall

Best-response polyhedron $\bar{P} = (x, v) \in R^M xR | x \geq vec0, B^T x \leq \vec{1}v, \vec{1}x = 1$ $\bar{Q} = (y, u) \in R^N xR | Ay \leq \vec{1}u, y \geq vec$

Facets are either own strategies (e.g. 4, 5 in the case of $\bar{Q}$) or strategies ot the other player (e.g. 1, 2, 3 in the case of $\bar{Q}$)

In particular, facets 1, 2 and 3 indicate best responses along with payoffs, e.g. for $y_4 \geq \frac{2}{3}$ a best response is 1. Facets 4 and 5 indicate when the respective own strategy has probability 0.

Def.: A point $(y, u) \in \bar{Q}$ has *label* $k \in M \cup N$ if the k^{th} inequality in the definition of $\bar{Q}$ is binding. For $k = i \in M$ this means $(Ay)_i = u$ (meaning strat i is a best response to y) For $k = j \in N$ this means $y_j = 0$ (meaning $y \notin supp(y)$)

Exp: $(\frac{2}{3}, \frac{1}{3}, 3) \in \bar{Q}$ has labels 1, 2.

Prop: A NE is a pair (x, y) of mixed strategies, such that with corresponding payoffs $u = x^T Ay$, $v = x^T By$, the pair $((x, v), (y, u)) \in \bar{P}x\bar{Q}$ is completely labeled, i.e. every label appears as a label of (x, v) or (y, u).

Simplify under the assumption that A, B^T are nonnegative and have no zero colums (nonzero payoffs). $\bar{P} = x \in R^M | x \geq vec0, B^T x \leq \vec{1}$ $\bar{Q} = y \in R^N | Ay \leq \vec{1}, y \geq vec0$

Remark: New Notions! x and y are now skaled (by u and v) versions of those from above. So they don't sum up to 1 anymore, but the ratio is still the same.

$\bar{P} \longleftrightarrow^{|:|} P \backslash \vec{0}$ $(x, v) \longrightarrow \frac{x}{v}$ $v = (1^T x)^{-1} \longleftarrow x$ $(x * v, v)$

This correspondence maps facets to facets and in partivcular labelled points to points with the same labels.

Cor.:

A NE is a completely labeled pair (x, y), where $x \in P \backslash \vec{0}, y \in Q \backslash \vec{0}$

Exp.:

Algo:

Equilibrium by vertex enumeration
Input: nondegenerate bimatrix game
Output: all NE of that game

1. NE = []
2. for all vertices $x \in P \backslash \vec{0}$ and all vertices $y \in Q \backslash \vec{0}$: if (x, y) is completely labeled: NE.append $(\frac{x}{1^T x}, \frac{y}{1^T y})$
3. return NE

For n=m:
$\sharp$ support pairs $\approx 4^n$
$\sharp$ vertex pairs $\approx 2.6^n$

Algorithm: LEMKE - HOWSON
Input: nondegenerate bimatrix game
Output: one(!) NE

1. $(x, y) \leftarrow (vec0, vec0) \in PxQ\sharp$ artificial equilibrium completely labeled
2. Pick a label $k \in M \cup N$ and drop it from (x, y). Call new vertex (x', y')
3. While (x', y') is not compzletely labeled alternatinglydrop duplicate labels and call resulting new verex (x', y').
4. return $(\frac{x'}{vec1^T x'}, \frac{y'}{vec1^T y'})$

Prop.: Lemke-Howson terminates and returns a NE.

Proof: For a fixed random choice k the traversal graph of visited vertices has only vertices of degree 1 (exuilibria) 2 (drop in P or in Q). The endpoints are completely labeled. Their number is even and subtracting the artifial one yields.

Cor.: A nondegenerate bimatrix game has an odd number of NE.