Cryptography, winter 2014/15
Hash functions

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In cryptography, hash functions are used, among other areas, in connection with signature schemes, where only the (short) hash values of (long) messages are signed. In the standard SHA-256, this looks as follows:

<table>
<thead>
<tr>
<th>length</th>
<th>arbitrary data</th>
<th>256 bits</th>
<th>512 bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>message $x$</td>
<td>$\rightarrow$ hash value $h(x)$</td>
<td>$\rightarrow$ signature $\text{sig}(h(x))$</td>
<td></td>
</tr>
</tbody>
</table>
Definition

Let $h: X \rightarrow Z$ be a mapping between two finite sets $X$ and $Z$. If $x \neq y$ are messages in $X$ with $h(x) = h(y)$, then $x$ and $y$ collide, and $(x, y)$ is a collision.
$X$

\[ X \]

\[ X \]

$Z$

\[ Z = h(x) = h(y) \]

$Z = h(x) = h(y)$
We consider three types of “attackers” $A$ on a hash function $h$.

(i) A collision finder $A$ takes no input and outputs either a collision $(x, y)$, or “failure”.

(ii) A second-preimage finder $A$ takes an input $x$ and outputs either some $y \in X$ that collides with $x$, or “failure”.

(iii) An inverter $A$ takes an input $z \in Z$ and outputs either some $x \in X$ with $h(x) = z$, or “failure”.

In any of these situation, we define the success probability $\sigma_A$ of $A$ as

$$\sigma_A = \text{prob} \left( A \text{ does not return “failure”} \right).$$
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The probability is taken over the internal random choices of $A$ and uniformly random choices $x \xleftarrow{} X$ in (ii), and $z \xleftarrow{} Z$ in (iii).
Definition

A function $\epsilon : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ is called *negligible* if for all $e \in \mathbb{N}$ there exists $N$ so that

$$\epsilon(n) \leq \frac{1}{n^e} \text{ for all } n \geq N.$$ 

If $\epsilon$ is not negligible, it is called *nonnegligible.*
Definition

Let $h = \{h_n\}$ be a family of hash functions. We call $h$

- collision resistant,
- second-preimage resistant,
- inversion resistant (or one-way),

if for all probabilistic polynomial-time

- collision finders $\mathcal{A}$,
- second-preimage finders $\mathcal{A}$,
- inverters $\mathcal{A}$,

respectively, for $h$ the success probability $\sigma_{\mathcal{A}}$ is negligible as function of $n$. 
Corollary

For a family $h$ of hash functions where the fraction $\frac{\#Z_n}{\#X_n}$ is negligible in $n$, we have

$$h \text{ collision resistant} \Rightarrow h \text{ second-preimage resistant} \Rightarrow h \text{ one-way}.$$
Birthday paradox

How many randomly chosen people have to be in a room to have a probability of at least 50% that two of them have the same birthday, assuming each birthday occurs with equal probability?
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How many randomly chosen people have to be in a room to have a probability of at least 50% that two of them have the same birthday, assuming each birthday occurs with equal probability?

Surprising answer:

23 people are sufficient!

Theorem:

We consider random choices, with replacement, among \( m \) labeled items. The expected number of choices until a collision occurs is \( O(\sqrt{m}) \).
For $e \in \mathbb{Z}$, we have

\[ g^e \text{ generates } G \iff \gcd(e, d) = 1. \]

Now we let $z \in G$ be also a generator, say $z = g^e$ with random $e \in \mathbb{Z}_d^\times$, and consider the function

\[ h_z : \mathbb{Z}_d \times \mathbb{Z}_d \to G, \quad (a, b) \mapsto g^a z^b. \]
We set $i = \lceil k/(\ell - n - 1) \rceil$, so that $x$ fits into $i$ blocks of $\ell - n - 1$ bits each, and $d = i(\ell - n - 1) - k < \ell - n - 1$ is the excess. We now pad $x$ by $d$ zeros, so that the length of this new word $\bar{x}$ is the multiple $i(\ell - n - 1)$ of $\ell - n - 1$. We split it as follows:

\[
\bar{x} = x_0 \quad x_1 \quad \ldots \quad x_{i-2} \quad x_{i-1} \quad x_i
\]

where each $x_j$ has $\ell - n - 1$ bits. We attach another word $x_i$ of $\ell - n - 1$ bits containing the binary representation of $d$, filled with leading zeroes.
Theorem

A collision for $h^*$ yields a collision for $h$. 
Figure: The $i$th round of SHA-256.
<table>
<thead>
<tr>
<th>function</th>
<th>semantic</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>The sum modulo $2^{32}$ of its two input integers</td>
</tr>
<tr>
<td>Ch($e, f, g$)</td>
<td><em>choice</em>: if $e$ then $f$ else $g$; equivalently $(e \text{ and } f)$ or (not $e$ and $g$)</td>
</tr>
<tr>
<td>$\Sigma_0(a)$</td>
<td>$\text{ROTR}^2(a) \oplus \text{ROTR}^{13}(a) \oplus \text{ROTR}^{22}(a)$</td>
</tr>
<tr>
<td>Maj($a, b, c$)</td>
<td><em>majority</em>: $(a \text{ and } b)$ or $(a \text{ and } c)$ or $(b \text{ and } c)$</td>
</tr>
<tr>
<td>$\Sigma_1(e)$</td>
<td>$\text{ROTR}^6(e) \oplus \text{ROTR}^{11}(e) \oplus \text{ROTR}^{25}(e)$</td>
</tr>
</tbody>
</table>

**Figure:** The internal SHA-256 functions. All except + are executed bitwise.
Special-purpose Bitcoin mining hardware computes blocks $M$ of a special structure such that

$$\text{SHA-256(SHA-256}(M))$$

is smaller than some pre-defined constant $B$. 
Figure: The SHA-3 sponge construction.
Figure: Collisions in the sponge construction.
Figure: The sponge claim.
One round of the SHA-3 $f$ function consists of five steps

$$R = \iota \circ \chi \circ \pi \circ \rho \circ \theta,$$

where $\iota$ is addition by some round specific constant.
Figure: A state of the SHA-3 $f$ function.
Figure: The step $\theta$ in the SHA-3 $f$ function

Compute

\[
 a[x][y][z] \leftarrow a[x][y][z] + \sum_{y'=0}^{4} a[x-1][y'][z] + \sum_{y'=0}^{4} a[x+1][y'][z-1].
\]
Compute

\[ a[x][y][z] \leftarrow a[x][y][z] - (t + 1)(t + 2)/2 \]

for some suitably selected \( 0 \leq t < 24 \).

**Figure:** The step \( \rho \) in the SHA-3 \( f \) function
Figure: The step $\pi$ in the SHA-3 $f$ function

Compute

$$a[x][y] \leftarrow a[x'][y']$$

for some suitably selected $x', y'$. 
Figure: The step $\chi$ in the SHA-3 $f$ function

Compute

$$a[x] \leftarrow a[x] + (a[x + 1] + 1)a[x + 2]$$