Cryptography, winter 2014/15

Hash functions

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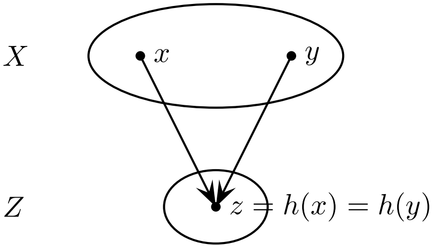
In cryptography, hash functions are used, among other areas, in connection with signature schemes, where only the (short) hash values of (long) messages are signed. In the standard SHA-256, this looks as follows:

length	arbitrary		256 bits		512 bits
data	message x	$\longrightarrow$	hash value $h(x)$	$\longrightarrow$	signature $\mathrm{sig}(h(x))$

Definition

Let $h\colon X \longrightarrow Z$ be a mapping between two finite sets X and Z. If $x \neq y$ are messages in X with $h(x) = h(y)$, then x and y *collide*, and (x, y) is a *collision*.


X
x
y
Z
$z = h(x) = h(y)$

We consider three types of "attackers" $\mathcal{A}$ on a hash function h.

(i) A *collision finder* $\mathcal{A}$ takes no input and outputs either a collision (x, y), or "failure".

(ii) A *second-preimage finder* $\mathcal{A}$ takes an input x and outputs either some $y \in X$ that collides with x, or "failure".

(iii) An *inverter* $\mathcal{A}$ takes an input $z \in Z$ and outputs either some $x \in X$ with $h(x) = z$, or "failure".

In any of these situation, we define the *success probability* $\sigma_{\mathcal{A}}$ of $\mathcal{A}$ as

$$\sigma_{\mathcal{A}} = \text{prob}\left(\mathcal{A} \text{ does not return "failure"}\right).$$

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The probability is taken over the internal random choices of $\mathcal{A}$ and uniformly random choices $x \xleftarrow{\text{⚄⚅}} X$ in (ii), and $z \xleftarrow{\text{⚄⚅}} Z$ in (iii).

Definition

A function $\epsilon\colon \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$ is called *negligible* if for all $e \in \mathbb{N}$ there exists N so that

$$\epsilon(n) \leq \frac{1}{n^e} \text{ for all } n \geq N.$$

If ϵ is not negligible, it is called *nonnegligible*.

Definition

Let $h = \{h_n\}$ be a family of hash functions. We call h

collision resistant,

second-preimage resistant,

inversion resistant (or *one-way*),

if for all probabilistic polynomial-time

collision finders $\mathcal{A}$,

second-preimage finders $\mathcal{A}$,

inverters $\mathcal{A}$,

respectively, for h the success probability $\sigma_{\mathcal{A}}$ is negligible as function of n.

Corollary

For a family h of hash functions where the fraction $\#Z_n/\#X_n$ is negligible in n, we have

h collision resistant $\Rightarrow$ h second-preimage resistant $\Rightarrow$ h one-way.

Birthday paradox

How many randomly chosen people have to be in a room to have a probability of at least 50% that two of them have the same birthday, assuming each birthday occurs with equal probability?

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Surprising answer:

23 people are sufficient!

Theorem:

We consider random choices, with replacement, among m labeled items. The expected number of choices until a collision occurs is $O(\sqrt{m})$.

For $e \in \mathbb{Z}$, we have

$$g^e \text{ generates } G \Longleftrightarrow \gcd(e,d) = 1.$$

Now we let $z \in G$ be also a generator, say $z = g^e$ with random $e \in \mathbb{Z}_d^\times$, and consider the function

$$h_z \colon \begin{array}{rcl} \mathbb{Z}_d \times \mathbb{Z}_d & \longrightarrow & G, \\ (a,b) & \longmapsto & g^a z^b \end{array}.$$

We set $i = \lceil k/(\ell - n - 1) \rceil$, so that x fits into i blocks of $\ell - n - 1$ bits each, and $d = i(\ell - n - 1) - k < \ell - n - 1$ is the excess. We now pad x by d zeros, so that the length of this new word $\bar{x}$ is the multiple $i(\ell - n - 1)$ of $\ell - n - 1$. We split it as follows:

$$\bar{x} = \underbrace{\overbrace{\boxed{}}^{\ell-n-1}}_{x_0}\ \underbrace{\overbrace{\boxed{}}^{\ell-n-1}}_{x_1}\ \cdots\ \underbrace{\overbrace{\boxed{}}^{\ell-n-1}}_{x_{i-2}}\ \underbrace{\boxed{\overbrace{}^{\ell-n-1-d}\,\overbrace{\ 0\ }^{d}}}_{x_{i-1}}\ \underbrace{\overbrace{\boxed{}}^{\ell-n-1}}_{x_i}$$

where each x_j has $\ell - n - 1$ bits. We attach another word x_i of $\ell - n - 1$ bits containing the binary representation of d, filled with leading zeroes.

$$\underbrace{\overbrace{\boxed{\;0\;}}^{n}\overbrace{\boxed{\;x_0\;}}^{\ell-n-1}\overbrace{\boxed{\;0\;}}^{1}}_{y_0}$$

$$\downarrow$$

$$\underbrace{\boxed{h(y_0)}\boxed{\;x_1\;}\boxed{\;1\;}}_{y_1}$$

$$\downarrow$$

$$\underbrace{\boxed{h(y_1)}\boxed{\;x_2\;}\boxed{\;1\;}}_{y_2}$$

$$\downarrow$$

$$\vdots$$

$$\downarrow$$

$$\underbrace{\boxed{h(y_{i-2})}\boxed{\;x_{i-1}\;}\boxed{\;1\;}}_{y_{i-1}}$$

$$\downarrow$$

$$\underbrace{\boxed{h(y_{i-1})}\boxed{\;x_i\;}\boxed{\;1\;}}_{y_i}$$

$$\downarrow$$

$$\boxed{h(y_i)}$$

Theorem

A collision for h^* yields a collision for h.

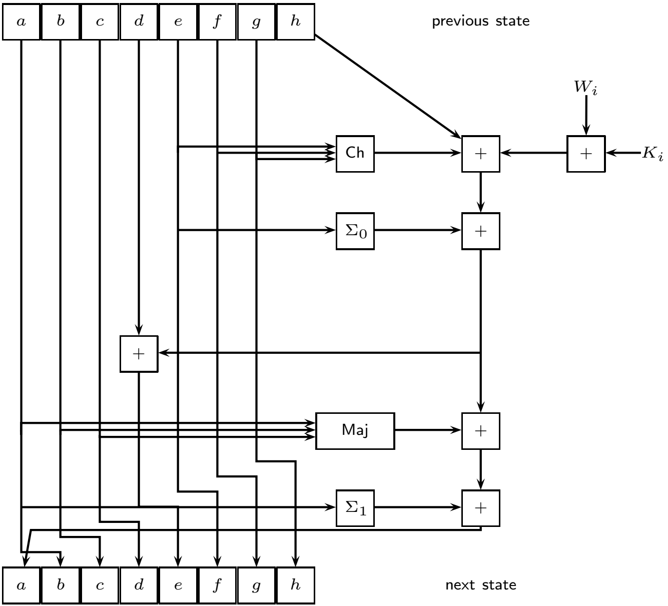

Figure: The ith round of SHA-256.

function	semantic
$+$	The sum modulo 2^{32} of its two input integers
Ch(e, f, g)	*choice*: if e then f else g; equivalenty (e and f) or (not e and g)
$\Sigma_0(a)$	$\mathsf{ROTR}^2(a) \oplus \mathsf{ROTR}^{13}(a) \oplus \mathsf{ROTR}^{22}(a)$
Maj(a, b, c)	*majority*: (a and b) or (a and c) or (b and c)
$\Sigma_1(e)$	$\mathsf{ROTR}^6(e) \oplus \mathsf{ROTR}^{11}(e) \oplus \mathsf{ROTR}^{25}(e)$

Figure: The internal SHA-256 functions. All except $+$ are executed bitwise.

Special-purpose Bitcoin mining hardware computes blocks M of a special structure such that

$$\text{SHA-256}(\text{SHA-256}(M))$$

is smaller than some pre-defined constant B.

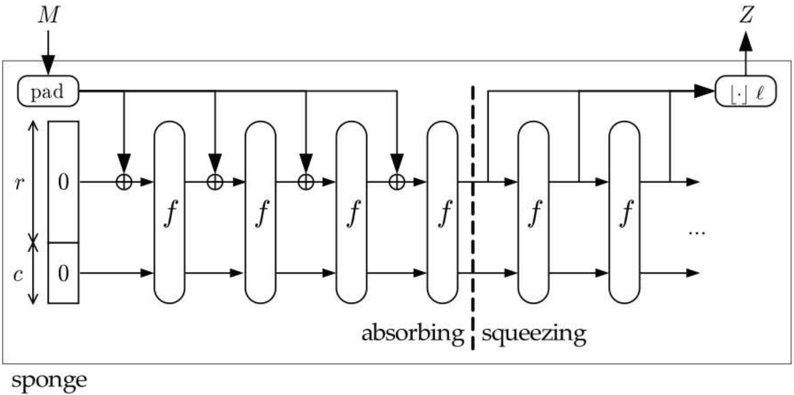

Figure: The SHA-3 sponge construction.

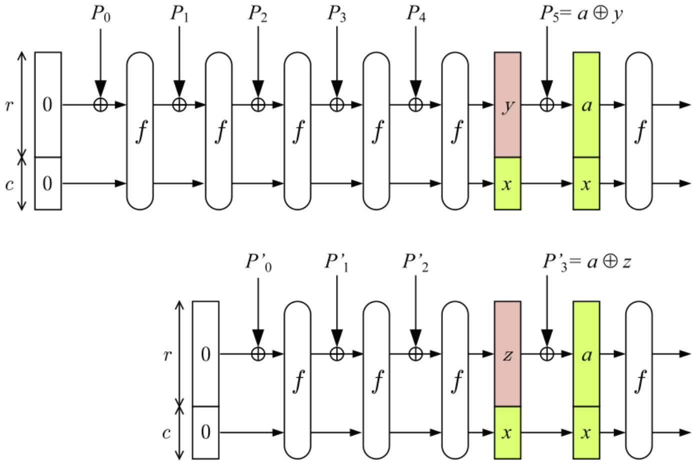

Figure: Collisions in the sponge construction.

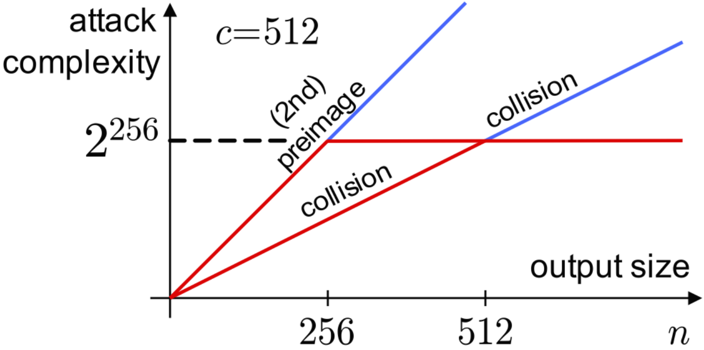

Figure: The sponge claim.

One round of the SHA-3 f function consists of five steps

$$R = \iota \circ \chi \circ \pi \circ \rho \circ \theta,$$

where ι is addition by some round specific constant.

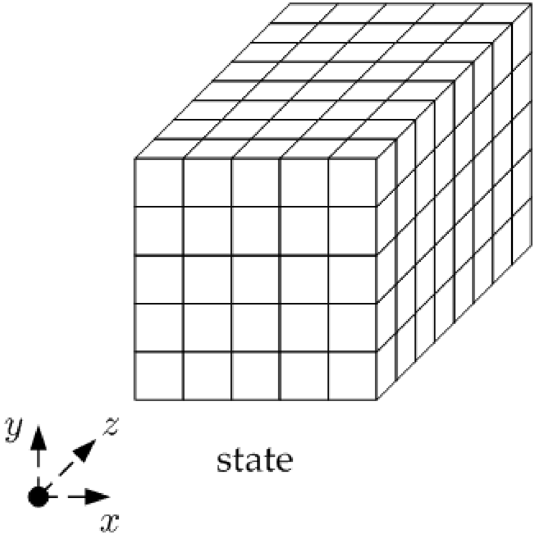

Figure: A state of the SHA-3 f function.

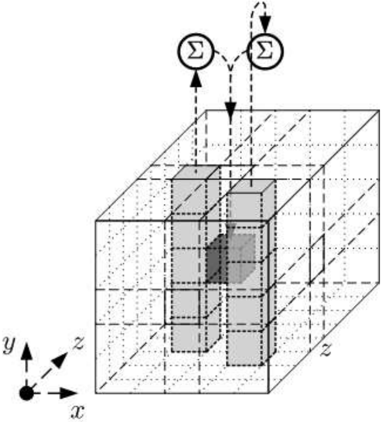

Figure: The step θ in the SHA-3 f function

Compute

$$a[x][y][z] \leftarrow a[x][y][z] + \sum_{y'=0}^{4} a[x-1][y'][z] + \sum_{y'=0}^{4} a[x+1][y'][z-1].$$

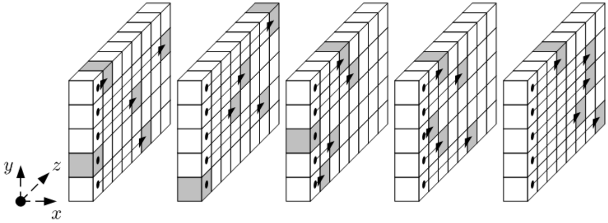

Figure: The step ρ in the SHA-3 f function

Compute

$$a[x][y][z] \leftarrow a[x][y][z - (t+1)(t+2)/2]$$

for some suitably selected $0 \leq t < 24$.

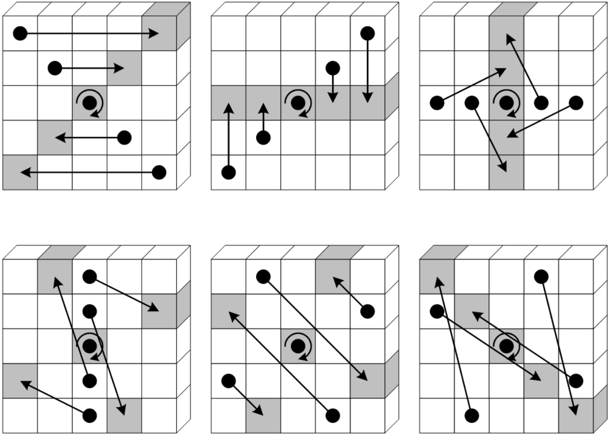

Figure: The step π in the SHA-3 f function

Compute

$$a[x][y] \leftarrow a[x'][y']$$

for some suitably selected x', y'.

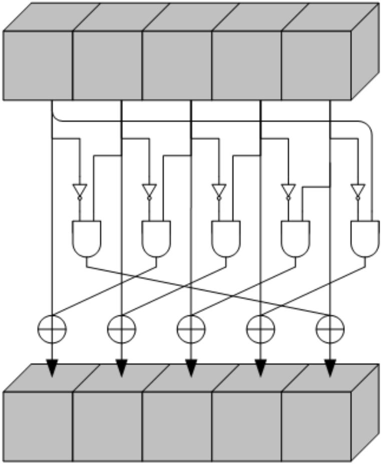

Figure: The step χ in the SHA-3 f function

Compute

$$a[x] \leftarrow a[x] + (a[x+1] + 1)a[x+2]$$